Nuclear reactions and nuclear fission

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1 Cross sections and reaction rates

- Consider a thin target of thickness $\mathrm{d}x$ and number density of targets n_t being bombarded by a beam of number density N_b moving at velocity v. Each target nucleus presents a target area σ – if one of the bombarding particles passes within the area σ, interaction (scattering, for example) occurs, otherwise not. Now if the beam covers an area A, the fraction of all the beam particles that will interact is the fraction of the beam area covered by targets:

$$\frac{\mathrm{d}N_\mathrm{b}}{N_\mathrm{b}} = \frac{\sigma n_\mathrm{t} A \mathrm{d}x}{A} = \sigma n_\mathrm{t} \mathrm{d}x.$$

  The beam intensity will therefore decrease with x as

$$N_\mathrm{b}(x) = N_{\mathrm{b},0} \exp(-n_\mathrm{t}\sigma x),$$

  and the number of particles removed from the beam is

$$N_\mathrm{sc}(x) = 1 - N_\mathrm{b}(x) = N_{\mathrm{b},0}[1 - \exp(-n_\mathrm{t}\sigma x)].$$

- The total scattering (or reaction) cross section describes the area that a target presents for any interaction with a particle in the beam. Experimentally, it is often interesting to look at the probability that a scattered particle ends up in some specific solid angle $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$. The number of particles $\Delta N_\mathrm{sc}(\theta, \phi)$ scattered into $\Delta\Omega$ is

$$\frac{\Delta N_\mathrm{sc}(\theta,\phi)}{N_{\mathrm{sc},0}} = n_\mathrm{t} x \frac{\mathrm{d}\sigma(\theta,\phi)}{\mathrm{d}\Omega}\Delta\Omega,$$

  where $\mathrm{d}\sigma(\theta,\phi)/\mathrm{d}\Omega$ is the *differential cross section*.

- The number of bombarding particles crossing unit area in $\mathrm{d}t$ is $N_\mathrm{b} v \mathrm{d}t$, so the number passing through σ in that time is $N_\mathrm{b} v \sigma \mathrm{d}t$. Hence the reaction rate per target nucleus is $N_\mathrm{b} v \sigma$ = flux $\times$ cross-section.

2 Resonance scattering and reactions

- The general theory of reactions between nuclei is too advanced for this course. Again we will resort to simple models that provide insight. In the case of a reaction between a bombarding particle and a target nucleus, we could imagine two rather different (idealized) ways in which reactions could occur. For definiteness, consider the following reaction:

$$^{2}_{1}\mathrm{D} + ^{39}_{19}\mathrm{K} \rightarrow ^{40}_{19}\mathrm{K} + \mathrm{p}.$$

  At one extreme, the reaction could proceed via a *compound nucleus* – the deuteron could be absorbed by the potassium, yielding a $^{41}_{20}\mathrm{Ca}^{*}$ nucleus in an excited state, which rattles around until it forgets about its origin and finally emits a proton. Alternatively, the reaction could occur *directly* – upon hitting the surface of the potassium, the neutron could become detached from the proton (the binding energy is only 2.2 MeV) and be absorbed by the nucleus, while the proton continues on its way (a stripping process); or the deuteron could be absorbed at the surface of the potassium, ejecting a proton as it enters (a knockout reaction). These different modes of reaction, which all end up with a proton leaving the potassium nucleus, may be distinguished by the rather different angular distributions of the ejected proton that are predicted.

- We have already seen that when a nucleus is bombarded by particles (e.g. neutrons, protons), the number of particles interacting with the nucleus or scattered out of the beam varies rapidly with energy, with local peaks in the number of particles scattered at specific energies associated with energy levels of the target nucleus. This is an example of *resonance scattering*. We may analyze this reaction using the compound nucleus model.

- It is found that resonance scattering occurs when the bombarding particle is captured by the nucleus, a process which is particularly likely to occur when the incoming particle carries just the right amount of kinetic energy to bring the nucleus to one of its excited states. This forms an excited compound nucleus state, which then after a while decays. Such an experiment is particularly easy to analyze when the incoming particle re-emerges (this is only one of the possible final states).

2.1 The resonant scattering cross section

- We have seen earlier that the lifetime of a state b is

$$\frac{1}{\tau} = \sum_{k} W_{kb} = \sum_{k} \frac{1}{\tau_{kb}}$$

  where W_{kb} is the transition rate (probability per second) for transitions from b to k, and τ_{kb} is lifetime for decay to a specific state k. We also define partial and total *widths* $\Gamma_{kb} = \hbar/\tau_{kb}$ and $\Gamma = \hbar/\tau$; the gammas have units of energy.

- We now consider a system initially in the form of a bombarding particle (e.g. a neutron) and a target nucleus I. The system forms a excited state of the compound nucleus X^{*}

and then decays to a two particle final state again. We will essentially consider these two steps separately, ignoring the effects of spin.

- Initially, the system is in a state ψ_1, which is one of a complete orthonormal set of (approximately stationary) states of the system, ψ_m. We have seen earlier (Lecture 2, p 5) that we may write the exact state of the system as a superposition of these stationary states

$$\Psi(t) = \sum_m a_m(t) e^{-iE_m t/\hbar} \psi_m,$$

where E_m is the energy of state m. We found that this leads to

$$i\hbar \dot{a}_n(t) = \sum_{m \neq n} H_{nm} a_m(t) e^{-i(E_m - E_n)t/\hbar}.$$

- Now we suppose that the system is initially in state i so $a_i(0) = 1$ and all other $a_m(0)$'s are zero. We are interested in the case in which the system forms the specific compound nucleus system c. For times not too long after $t = 0$ when the incoming neutron approaches the target, the probability amplitude (while $a_i(t)$ is still nearly 1) of finding the system in c satisfies approximately

$$i\hbar \dot{a}_c = H_{ci} e^{-i(E_i - E_c)t/\hbar} - i(\Gamma/2) a_c.$$

Here the first term describes the probability of making the transition into state c and the second lumps together all the ways of leaving c that lead to the mean lifetime Γ of this state.

- This equation may be rewritten as

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}(a_c e^{\Gamma t/2\hbar}) = H_{ci} e^{-i(E_i - E_c + i\Gamma/2)t/\hbar}.$$

Integrating from 0 to $t >> \hbar/\Gamma$ and dividing by $i\hbar e^{\Gamma t/2\hbar}$, we find the probability that the system is in state c to be

$$|a_c(t)|^2 = \frac{|H_{ci}|^2}{(E_i - E_c)^2 + \Gamma^2/4},$$

and recalling that the decay rate of c into the final state (channel f) is $1/\tau_f = \Gamma_f/\hbar$, the decay rate to f may be written as

$$|a_c(t)|^2/\tau_f = \frac{|H_{ci}|^2}{(E_i - E_c)^2 + \Gamma^2/4} \left(\frac{\Gamma_f}{\hbar}\right).$$

- Now the incoming particle is described by a wave function $\exp i\mathbf{k} \cdot \mathbf{r}/\sqrt{V}$ in the centre of mass coordinate system, and V is the (artificial) volume of the box the system is in. The particle *flux* is given by the particle density times the beam velocity, $V^{-1}v$, and so the cross section for transitions from channel i to channel f is given by

$$V^{-1} v \sigma_{if} = |a_c(t)|^2/\tau_f.$$

- From Fermi's golden rule (Lecture 2, p 7), the width of the decay from state c to the *initial* state i which has energy E is

$$\Gamma_i(E) = \frac{\hbar}{\tau_i} = 2\pi |H_{ic}|^2 n_i(E)$$

where $n_i(E)$ is the density of states around i, which we have seen is $n_i(E)\mathrm{d}E = (V/2\pi^3)4\pi k^2(\mathrm{d}k/\mathrm{d}E)\mathrm{d}E$. Since $E = \hbar^2k^2/2m$, $(\mathrm{d}E/\mathrm{d}k) = \hbar^2k/m = \hbar v$.

- Finally putting everything together, we get

$$\sigma_{if} = \frac{V}{v}\frac{|H_{ci}|^2}{(E_i - E_c)^2 + \Gamma^2/4}\left(\frac{\Gamma_f}{\hbar}\right) = \frac{V}{v}\frac{1}{2\pi\hbar}\frac{1}{n_i(E)}\frac{\Gamma_i\Gamma_f}{(E_i - E_c)^2 + \Gamma^2/4}$$

so

$$\sigma_{if} = \frac{\pi}{k^2}\frac{\Gamma_i\Gamma_f}{(E_i - E_c)^2 + \Gamma^2/4}.$$

This important result is often called the Breit-Wigner formula.

- Points to notice:

1. The cross section σ_{if} is peaked around $E_i = E_c$ and falls to half its peak value at $\pm\Gamma/2$ (hence the term "width" for Γ).
2. The width of the resonant peak depends on the lifetime of the compound nucleus intermediate state; measurement of the width provides an estimate of this lifetime.
3. The derivation ignores the contribution of non-resonant scattering, which contributes a cross section of about the geometrical value, $\sigma_{\mathrm{nr}} \approx 4\pi R^2$ far from resonances.
4. This expression predicts that the maximum cross section can be of order $\pi/k^2 = \pi\lambda^2 = \pi\hbar^2/p^2$, where λ is the de Broglie wavelength of the incoming neutron. For low energy bombarding particles, this can be quite a lot larger than the (non-resonant) geometrical cross section.
5. The non-resonant scattering term interferes with the resonant one, so that the cross section drops below the geometrical value on one side of a resonance.

2.2 Applications to experiments

- The cross section for interaction of a neutron beam with a target does not fall off rapidly with energy because there is no Coulomb barrier to surmount. Neutrons can be absorbed by a nucleus even at $E = 0$. At sufficiently low energies, σ usually *rises* as $E \to 0$, so that the cross section follows a "$1/v$" law.

- This can be made plausible for the case where the low energy limit is not too far from a resonance, so that we may use the Breit-Wigner formula. Recall from Fermi's golden rule that Γ_i is proportional to the density of states times a matrix element. If we suppose that the matrix element does not change much at low energy, then Γ_i depends

on energy as $n_i(E) = \mathrm{d}\mathcal{N}(E)/\mathrm{d}E$ which is proportional to $E^{1/2}$ or k. Taking this dependence into account, the variation of σ for neutron interaction goes as

$$\sigma \approx \frac{1}{k}\frac{\text{const}}{E_0^2 + \Gamma^2/4},$$

which varies as $1/k$ or $1/v$ as $E \to 0$.

- For reactions of charged particles with other nuclei, one must take the Coulomb repulsion into account. We may get an idea of how this changes things by looking at the specific case of bombardment of a target by alpha particles. We use the Breit-Wigner formula, which may be written as

$$\sigma_{if}(E) = \frac{\pi\hbar^3}{2mE}\frac{(\Gamma_i/\hbar)\Gamma_f}{(E_i - E_c)^2 + \Gamma^2/4}$$

by recalling that $E = \hbar^2k^2/2m$. Now we have seen in the case of alpha *decay* that the probability per second of decay for energies small compared to the Coulomb barrier is $(1/\tau_0)\exp -G(E)$. This is also equal to $\Gamma_i/\hbar$, and making this substitution and letting E become much smaller than the energy E_c of the lowest resonance, we find

$$\sigma_{if}(E) \approx \frac{\pi\hbar^3}{2m\tau_0}\frac{\Gamma_f}{E_c^2 + \Gamma^2/4}\frac{e^{-G(E)}}{E}.$$

This shows that the cross section for interaction goes to 0 exponentially as $E \to 0$.

- This result is an example of more general behaviour of the interaction cross section for charged particles. We see that our result may be written as

$$\sigma(E) = \frac{1}{E}S(E)e^{-G(E)}$$

where $S(E)$ is approximately constant for small enough E. This behaviour is quite general, and it is common to describe low-energy experimental data by specifying $S(E)$. This is particularly useful in stellar astrophysics which often needs reaction cross sections at lower energies than those at which they can be measured.

3 Nuclear fission as a power source

- One extremely important aspect of nuclear physics is its application as a source of energy for society. At present about 1/6th of world energy generation comes from nuclear power. In Ontario, almost half the electric power generation comes from nuclear power plants.

- One expects that fission of very heavy elements could provide a good energy source because of the decline of the average binding energy per nucleon beyond iron from a peak of somewhat more than 8.5 MeV per nucleon to about 7.5 MeV per nucleon. This means that if a nucleus with $A = 240$ fissions, about 1 MeV per nucleon will

be released, providing in total roughly 200 MeV per fission. Thus fissioning 1 kg of U should release about 8×10^{13} J (about 20 million kWh), compared to the energy release by burning (oxidizing) 1 kg of natural gas which releases roughly 10^8 J or about 2 kWh.

- The most suitable naturally occurring fuel is uranium, which is available in adequate deposits for mining in regions of old continental shield such as northern Canada. It occurs primarily in two forms, $^{235}_{92}$U (0.7%) and $^{238}_{92}$U (99.3%). For uranium ore, the natural rate of fission is completely negligible, but U may be made to fission by neutron capture, and as each fission results in neutron-rich daughter nuclei which release a couple of neutrons that may in turn be captured, the possibility of a self-sustaining reaction exists.

- When a uranium nucleus captures a neutron, the nucleon enters the nucleus in an excited state of energy equal to the binding energy of the last neutron plus the (centre-of-mass) kinetic energy of the neutron at the instant of capture. If this excitation energy is larger than the Coulomb barrier height of 5 to 6 MeV, the nucleus will fission immediately; otherwise it may well de-excite radiatively without fissioning. Here there is an important difference between the two natural isotopes of uranium; $^{235}_{92}$U changes from an odd-even to an even-even nucleus on neutron capture, while $^{238}_{92}$U makes the opposite change. Because of the symmetry energy term in the semi-empirical binding energy formula, the new neutron has higher excitation energy in $^{235}_{92}$U (about 6.46 MeV even without any kinetic energy) than in $^{238}_{92}$U (only 4.78 MeV plus kinetic energy). This difference is just large enough that $^{235}_{92}$U fissions even on capture of a thermal neutron, but $^{238}_{92}$U must capture a neutron having at least 1.4 MeV of energy to fission.

- Another important difference between the two natural isotopes of uranium is that – apart from resonances – the interaction cross section for fission is roughly constant around 10 barns from thermal energies to several MeV for $^{238}_{92}$U, while the fission cross section of $^{235}_{92}$U is more than 100 barn and rising below about 1 eV. Thus thermal neutrons are more easily captured in $^{235}_{92}$U, by a factor large enough to almost make up for the low abundance of this isotope.

- On fission, the two highly excited fission fragments very quickly boil off on average about $\nu = 2.5$ neutrons, plus about $\nu_{\rm d} = 0.02$ more after some beta-decays and a delay of some seconds. The total energy release, mostly in the form of the kinetic energy of the fission fragments, is about 205 MeV. (Some 12 MeV escapes with neutrinos from the beta decays.)

- The results of fission followed by release of neutrons which cause other fissions depends sensitively on the number of neutrons released and captured per fission. If q is th probability of capture, then the change in the number $n(t)$ of free neutrons in $\mathrm{d}t$ is $n(t + \mathrm{d}t) = n(t) + (\nu q - 1)n(t)(\mathrm{d}t/t_{\rm p})$ where $t_{\rm p}$ is the time-scale for a neutron to either react or somehow escape from the system. This equation leads to

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\nu q - 1}{t_{\rm p}} n(t)$$

with solution

$$n(t) = n(0)e^{(\nu q - 1)t/t_{\rm p}}.$$

Thus the number of reaction either falls off rapidly or increases catastrophically depending on whether $\nu q < 1$ (uranium ore situation) or $\nu q > 1$ (nuclear bomb). Only for $\nu q = 1$ – the reaction rate balanced on a knife-edge – can we get useful power generation.

- A typical fission neutron has an energy of about 2 MeV. The total cross section for such neutrons is about 7 barns, so the mean free path $l \approx 1/(n_{\rm U}\sigma)$ where ($n_{\rm U}$ is the number density of uranium atoms in the pure metal, about 4.8×10^{28} nuclei per cubic meter. The mean free path is thus about 3 cm. For pure $^{235}_{92}$U, the scattering cross section is about 5 times larger than the cross section for fission, so the neutron typically goes several cm before causing another fission, requiring about $t_{\rm p} \approx 10^{-8}$ s for this. Thus if the reaction is exponentially increasing, the time-scale for explosion is less than 1 microsecond once a critical mass (about 50 kg of $^{235}_{92}$U, or a sphere of about 17 cm diameter) has been assembled.

- Now consider the possibility of a power reactor. With natural uranium fuel, a 2 MeV neutron can cause fission in either isotope, but the probability of scattering (inelastically) is about 20 times higher than that of causing fission in $^{238}_{92}$U, so the neutron quickly drops in energy below the fission threshold. Thus $^{235}_{92}$U is not directly usable as a fuel. Now as the neutron energy drops, it has a fair probability of being absorbed by $^{238}_{92}$U without causing fission. (This produces the useful fuel – and bomb-making material – plutonium $^{239}_{94}$Pu after two beta decays, thus leading to the possibility of a breeder reactor, a nuclear reactor that creates more fuel than it uses.)

- In the usual thermal reactor, the drop in energy is deliberately controlled in such a way as to lead to fission of the $^{235}_{92}$U fuel in the reactor and also to control the power output. The fuel is in the form of thin rods embedded in a substance (the *moderator*) of low mass number and low neutron absorption cross section. The neutrons quickly escape from the fuel rod (leading to low probability of capture by $^{238}_{92}$U) and lose their energy by elastic collisions with the moderator until most have reached the thermal energy of the reactor, about 0.1 eV. Because of the large capture cross section for such neutrons by $^{235}_{92}$U, enough neutrons are captured to make the reaction self-sustaining. The moderator is usually carbon or heavy water D_2O (the high neutron capture cross section of normal H makes ordinary water unsuitable for a natural fuel reactor). Control is achieved by inserting boron or cadmium rods into the reactor core; the large capture cross section of B or Cd for neutrons leads to a decrease in the neutron flux and slows the reactor down.