Alpha decay, fission, and nuclear reactions

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1 Energy release in alpha-decay

- Consider a nucleus which is stable against decay by proton or neutron emission – the least bound nucleon still has (say) several MeV of binding energy. This nucleus may nevertheless still be able to decay by using the fact that if it can emit an *alpha-particle* ($^4_2$He), it can use the *binding energy* of the alpha to supply the energy needed for the escape. That is, the available energy release in alpha decay is

$$Q(Z, A) = B(Z-2, A-4) - B(Z, A) + 28.3 \ \ \text{MeV},$$

  where the third (energy-contributing)term arises from the energy made available by forming an alpha.

- Using the semi-empirical mass formula and the $N-Z$ relationship for maximum binding, one finds that essentially all nuclei with $Z > 66$ are unstable to alpha-decay. However, the instability is so slow as to be almost unobservable up to about $Z = 83$.

- Alpha-decay is usually extremely slow even when it is energetically permitted because of the existence of a potential barrier between the bound nucleus and the free world outside. The effect of electrostatic repulsion is to lift the potential well of the protons relative to that of the neutrons: the lip of the proton well is higher than that of the neutrons. The fermi energy of both kinds of particles is about the same (this is why $N > Z$ for large A, of course) so the protons are more tightly bound. However, in heavy nuclei the energy available from forming an alpha particle is enough to make alpha-decay energetically possible – provided the alpha can get far enough from the nucleus to have positive total energy.

- There is thus a barrier penetration problem. Quantum mechanics makes this possible. How can we estimate the probability of alpha-decay without knowing the details of the nuclear force and the states of the nucleons in the nucleus?

- We will treat the problem of forming the alpha particle for later. Suppose that we have somehow momentarily formed an alpha on the outside surface of a nucleus. What is the probability that it will be able to escape? Outside the nucleus, the Schroedinger equation for the radial wave function is

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(ru) + \left[\frac{2Z_\mathrm{d}e^2}{(4\pi\varepsilon_0)r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Qu,$$

where Z_d is the charge on the daughter nucleus and m is the reduced mass of the alpha particle.

- Consider the case $l = 0$ (actually the angular momentum term is usually small compared to the electrostatic term). Now if the Coulomb term were a constant, V_0, the solutions $f(r) = ru(r)$ of the radial wave equation would be

$$f(r) = e^{\pm ikr}, \quad Q > V_0, \quad k^2 = \frac{2m}{\hbar^2}(Q - V_0)$$

for an alpha of positive total energy, and

$$f(r) = e^{\pm Kr}, \quad Q < V_0, \quad K^2 = \frac{2m}{\hbar^2}(V_0 - Q)$$

for particles of negative total energy. This suggests that we should look for solutions of the actual problem of the form $f(r) = \exp[\phi(r)]$. Substituting, $\phi(r)$ satisfies

$$\frac{\hbar^2}{2m}\left[\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^2\right] = \left(\frac{2Z_\mathrm{d}e^2}{(4\pi\varepsilon_0)r} - Q\right).$$

In the case of a constant potential, we see that $\mathrm{d}^2\phi/\mathrm{d}r^2 = 0$. Let's try to solve our equation assuming that this term is small compared to $(\frac{\mathrm{d}\phi}{\mathrm{d}r})^2$ and may be neglected. The equation simplifies to

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \pm\left[\frac{2m}{\hbar^2}\left(\frac{2Z_\mathrm{d}e^2}{(4\pi\varepsilon_0)r} - Q\right)\right]^{1/2}$$

so

$$\phi(r) = \pm\int\left[\frac{2m}{\hbar^2}\left(\frac{2Z_\mathrm{d}e^2}{(4\pi\varepsilon_0)r} - Q\right)\right]^{1/2}\mathrm{d}r.$$

- Thus beyond the classical turning point r_c, where the quantity inside the radical is negative, we can write the solution approximately as

$$f(r) = A\exp[+i\int_{r_\mathrm{c}}^{r} k(r)\mathrm{d}r] + B\exp[-i\int_{r_\mathrm{c}}^{r} k(r)\mathrm{d}r],$$

where

$$k(r) = \left[\frac{2m}{\hbar^2}\left(Q - \frac{2Z_\mathrm{d}e^2}{(4\pi\varepsilon_0)r}\right)\right]^{1/2}.$$

Since the full time-dependent wave in this region is multiplied by $\exp(-i\omega t)$, we see that outside the classical turning point r_c the solution is a sum of an outgoing wave (with coefficient A) and an incoming wave (with coefficient B). We are concerned only with escaping alpha particles, so we may set $B = 0$.

- In the classically forbidden region between the surface of the nucleus at r_s and the turning point, the solution is approximately

$$f(r) = C\exp[+\int_r^{r_\mathrm{c}} K(r)\mathrm{d}r] + D\exp[-\int_r^{r_\mathrm{c}} K(r)\mathrm{d}r],$$

where

$$K(r) = \left[\frac{2m}{\hbar^2}\left(\frac{2Z_\mathrm{d}e^2}{(4\pi\varepsilon_0)r} - Q\right)\right]^{1/2}.$$

In this region the wave function is an exponential function which declines from inside r_c towards the turning point (multiplied by C), plus a term which exponentially declines as one moves inward from the turning point (with coefficient D). Again, it seems clear that the term with D must represent particles trying to penetrate from the outside, while the C term represents particles penetrating from inside. Consequently we put $D = 0$.

- Now to obtain the full wavefunction of the escaping alpha particle, we would have to join the wavefunction in the nucleus (which we don't know – how many of the nucleons are in the form of alphas at once, and where?) onto the declining exponential solution in the forbidden region, and join that onto the outgoing wave solution beyond r_c, by using the continuity of the wavefunction and its first derivative.

- However, we can already get a very interesting result by simply considering the decline of the wave function through the classically forbidden region. Suppose that an alpha particle forms at the surface of the nucleus, at r_s. Then the *probability density* of finding the alpha at the turning point r_c is

$$\frac{4\pi r_\mathrm{c}^2|u(r_\mathrm{c}|^2}{4\pi r_\mathrm{s}^2|u(r_\mathrm{s}|^2} = \frac{4\pi|f(r_\mathrm{c})|^2}{4\pi|f(r_\mathrm{s})|^2} = \frac{|f(r_\mathrm{c})|^2}{|f(r_\mathrm{s})|^2},$$

where $u(r)$ is the radial factor in the original separated wave equation. So now if we can simply calculate the right hand side of this equation, we can see how the probability of escape varies with the conditions around the nucleus that we do know about – mainly its repulsive charge, and the distance between r_s and r_c (which is determined by the energy available for the reaction).

- Call

$$\frac{|f(r_\mathrm{c})|^2}{|f(r_\mathrm{s})|^2} = e^{-G},$$

where from the equations above

$$G = 2\int_{r_\mathrm{s}}^{r_\mathrm{c}} K(r)\mathrm{d}r = 2\int_{r_\mathrm{s}}^{r_\mathrm{c}}\left[\frac{2m}{\hbar^2}\left(\frac{2Z_\mathrm{d}e^2}{(4\pi\varepsilon_0)r} - Q\right)\right]^{1/2}.$$

Since $r_\mathrm{c} = 2Z_\mathrm{d}e^2/(4\pi\varepsilon_0 Q)$ we may write this as

$$G = 2\left(\frac{2mQ}{\hbar^2}\right)^{1/2}\int_{r_\mathrm{s}}^{r_\mathrm{c}}\left(\frac{r_\mathrm{c}}{r} - 1\right)^{1/2}\mathrm{d}r.$$

- Substitute $r = r_\mathrm{c}\cos^2(\theta)$, and the integral becomes

$$G = 2r_\mathrm{c}\left(\frac{2mQ}{\hbar^2}\right)^{1/2}\int_0^{\theta_0} 2\sin^2\theta\mathrm{d}\theta = 2r_\mathrm{c}\left(\frac{2mQ}{\hbar^2}\right)^{1/2}(\theta_0 - \sin\theta_0\cos\theta_0)$$

where $\theta_0 = \cos^{-1}[(r_\mathrm{s}/r_\mathrm{c})^{1/2}]$. Thus finally

$$G = \frac{\pi}{\hbar c}\left(\frac{2Z_\mathrm{d}e^2}{4\pi\varepsilon_0}\right)\left(\frac{2mc^2}{Q}\right)^{1/2}\mathcal{G}(r_\mathrm{s}/r_\mathrm{c}),$$

where we define the dimensionless function

$$\mathcal{G}(x) = \frac{2}{\pi}\left[\cos^{-1}(\sqrt{x}) - \sqrt{x(1-x)}\right]$$

which runs from 1 at $x = 0$ to 0 at $x = 1$. When the available energy Q is low, r_c is large and $\mathcal{G}(r_\mathrm{s}/r_\mathrm{c})$ approaches 1.

- Now how can we use this result? If the creation rate of alphas at the surface of the nucleus is τ_0^{-1}, then the probability per second of one of these escaping to the classical turning radius is $\tau_0^{-1}e^{-G}$, and the mean life of the unstable parent nucleus is the inverse of this, or $\tau = \tau_0 e^G$. This result allows us to predict the *ratios* of the lifetimes for decay of various alpha-unstable nuclei in a series. We will need to guess (or choose appropriately) the value τ_0, but then the variation of τ from one nucleus to another can be computed.

- Consider the series of alpha decays starting with $^{238}_{92}$U. The measured lifetimes of the various nuclei in this series are given in C & G, Table 6.1. The measured lifetimes range over 20 powers of ten. Can we really account for this enormous variation with our simple theory?? Let us take $\tau_0 = 7.0 \times 10^{-23}$ s (this choice will turn out to work well for almost all the decays in this series, and is approximately the nuclear crossing time), and note that numerically

$$G = 1.979 Z_\mathrm{d}\sqrt{m/Q}\,\mathcal{G}(r_\mathrm{s}/r_\mathrm{c}),$$

where m is the reduced mass of the alpha particle measured in atomic mass units (it is a number close to 4.0), and Q is the energy release in MeV (often a number in the range of 3 to 10). Now for a typical decay, we find the G is of the order of 10^2, so the tunneling factor e^G is of the order of 10^{40}. Furthermore, a rather small change in Q and $\mathcal{G}$ can change G by a factor of roughly 2 – thus changing the the quantity e^G by something like 10^{20}! There is a huge increase in mean lifetime as the available energy drops from, say, 6 or 8 MeV to around 4 MeV.

2 Fission

- Fission can occur if enough energy is available that the resulting two fragments have less total energy than the original. This is particularly facilitated if the parent nucleus is supplied with some extra energy (e.g. by proton bombardment, neutron capture, etc). However, for heavy nuclei fission is possible from the ground state. We may use the semi-empirical mass formula to discover the general conditions for fission to occur. Suppose we consider the specific possibility of symmetric fission $(Z, A) \to 2(Z/2, A/2)$. We ignore the pairing energy, so that the only terms in the formula that change are those for surface and for Coulomb self-energy:

$$\Delta B = -bA^{2/3}[2(1/2)^{2/3} - 1] - \frac{dZ^2}{A^{1/3}}[2(1/2)^{5/3} - 1].$$

The first term is positive, the second negative, so we see that for

$$\frac{Z^2}{A} > \frac{b(2 - 2^{2/3})}{d(2^{2/3} - 1} \approx 18$$

fission is energetically possible. This condition applies for Z larger than about 42. However, the tunneling factor is so small that spontaneous fission is only seen among the highest mass elements.

- For really massive nuclei, fission is possible without a tunnel problem. To see this, think about how the fission process might actually occur, by forming first a sort of ellipsoid of revolution which then becomes something like a peanut and finally splits. To see how this event changes the total binding energy of the initial nucleus, we need to look at the surface area and Coulomb terms again. C & G report that in the ellipsoidal stage when the eccentricity is ϵ, the surface area changes to

$$S(\epsilon) = 4\pi R^2(1 + \frac{2}{3}\epsilon^2 - \frac{52}{105}\epsilon^3),$$

so that the corresponding term in the binding energy should change to

$$-bA^{2/3}(1 + \frac{2}{3}\epsilon^2 - \frac{52}{105}\epsilon^3).$$

Similarly, the change to the Coulomb term will be

$$-d\frac{Z^2}{A^{1/3}}(1 - \frac{1}{5}\epsilon^2 + \frac{4}{21}\epsilon^3).$$

- The total change in binding energy as ϵ changes from 0 is the sum of these two terms:

$$\left(\frac{2}{5}bA^{2/3} - \frac{1}{5}d\frac{Z^2}{A^{1/3}}\right)\epsilon^2 - \left(\frac{52}{105}bA^{2/3} - \frac{4}{21}d\frac{Z^2}{A^{1/3}}\right)\epsilon^3.$$

The coefficient of the ϵ^2 term is negative for

$$\frac{Z^2}{A} > \frac{2b}{d} = 51,$$

so the energy *decreases* as soon as ϵ differs from 0, and fission is completely unrestrained by any barrier. There is thus an absolute instability to fission for Z greater than about 144 (using the general $Z - A$ relation for beta stability).

- The results above suggest that the value of Z^2/A is the key parameter controlling the tendency towards fission, and empirically there is a definite relation, with shorter lifetimes found for nuclei with larger values of this quantity.

- The nuclei produced in fission are overly neutron rich, and typically boil off a couple of neutrons before settling down towards the stability valley by beta-decays. These prompt neutrons are quite important in controlled fission for power production, as we will see later.

- Although the mass formula predicts a maximum energy release when fission produces two nearly identical nuclei, real fission generally leads to one nucleus that is considerably more massive than the other, probably because of shell structure effects.

3 Measurement of energies of excited states

- We have seen from the density of states argument that nuclei should possess excited states in which one or several nuclei are in higher single-particle orbitals, or in which collective motions (vibrations, rotations) give the system a higher internal energy than the ground state. The excited energy levels may be detected and their energies measured by a number of methods, for example by striking a target material with a beam of protons of known energy and measuring the energies of outcoming protons at various scattering angles. Some interactions are elastic, but others leave the target nucleus in an excited state by extracting energy from the deflected proton, and measurement of the energy of the outgoing proton can reveal the energy loss that has occurred.

- In a collision, momentum will be completely conserved, generally between only the bombarding and target particles. If the incoming proton had momentum p_i and emerges at angle θ with p_f, the bombarded nucleus will have momentum along the beam of $(p_\mathrm{i} - p_\mathrm{f}\cos\theta)$ and momentum $p_\mathrm{f}\sin\theta$ normal to the beam. In the non-relativistic limit, the difference between the initial energy of the proton and the sum of the final energies of the proton and of the target is therefore

$$\Delta E = \frac{p_\mathrm{i}^2}{2m_\mathrm{p}} - \frac{p_\mathrm{f}^2}{2m_\mathrm{p}} - \frac{p_\mathrm{i}^2 + p_\mathrm{f}^2 - 2p_\mathrm{i}p_\mathrm{f}\cos\theta}{2m_A},$$

  where m_A is the mass of the target nucleus. This may be rewritten in terms of initial and final proton energies as

$$\Delta E = E_\mathrm{i}\left(1 - \frac{m_\mathrm{p}}{m_A}\right) - E_\mathrm{f}\left(1 + \frac{m_\mathrm{p}}{m_A}\right) + \frac{2m_\mathrm{p}}{m_A}\sqrt{E_\mathrm{i}E_\mathrm{f}}\cos\theta.$$

- If no energy is transferred into excitation of internal states of the target, ΔE should be 0, but the outgoing proton will no longer have as high an energy as it initially did because of transfer to the target. For a particular scattering angle θ the equation above may be solved for the value of E_f of elastically scattered protons.

- However, one finds in experiments that not all the scattered protons at a particular θ have the energy predicted for $\Delta E = 0$. Usually several smaller proton energies are also observed, due to absorption by the nucleus of specific quanta of energy from the scattered protons.

- Rare isotopes of an element may be studied, for example, by techniques like deuteron stripping. In this kind of experiment, the target is bombarded with deuterons. Sometimes a nuclear reaction occurs, for example leaving behind a particle in the target nucleus, so that one may study the states of the new isotope.

- From such experiments one may obtain *energy level diagrams* for various nuclei. By examining the angular distribution of outcoming particles, information about the angular momentum and parity of the excited states may also be gotten.

- A striking result is that the energy level diagrams of mirror nuclei – pairs in which the neutron number N of one is the proton number Z of the other, and vice versa (e.g. ${}^{11}_{5}\mathrm{B}$ and ${}^{11}_{6}\mathrm{C}$) – are very similar, both in energy spacing and in other properties such as parity and spin.

- In general, heavy nuclei have more states at a given energy above ground than light ones do, and the higher one goes above the ground state, the more closely the states are spaced.

4 Electromagnetic decay of excited states

- When a nucleus is raised to an excited state, for example as the result of a beta-decay in which the daughter nucleus is left in an upper energy level, the nucleus will spontaneously drop to the ground state. One important way in which this happens is by emission of one or more photons (note that such nuclear emissions are called gamma rays, while photons produced in the surrounding electron cloud, which may have similar energies, are called X-rays). Measurement of gamma ray energies provide a valuable means of studying the excitation energies of nuclear excited states.

- The lifetime before gamma emission depends very strongly on the difference between the total angular momenta of the two states. For each increase in ΔJ above 1, the lifetime increases by a factor of the order of 10^4 or more, from a minimum which may be a short as 10^{-16} s. Thus states in which the photon must carry off several units of $\hbar$ of angular momentum may have lifetimes of hundreds of years against radiative decay. Such long-lived excited states are called *isomeric states*.

- A related way in which a nucleus may lose energy by an electromagnetic transition is what is known as *internal conversion*, in which the nuclear energy is transferred directly to an orbitting electron, which is then ejected carrying the energy difference between the two nuclear states, less the initial binding energy of the electron before ejection.

- It is not uncommon for beta-decay to leave the daughter nucleus in an excited state. This is probably the most common mechanism by which excited nuclei are produced naturally on earth. After the beta-decay, the excited daughter will decay, often by emission of one or more gammas, or by internal conversion. Again, the energies of emitted photons or electrons provide direct evidence about the energy levels of the daughter nucleus.