Nuclear sizes and ground states

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1 Nuclear dimensions

- Nuclear size may be probed with *scattering experiments*, such as Rutherford's bombardment of a gold foil with alpha-particles which established the nuclear model of the atom, or electron scattering. Bombarding particles must have de Broglie wavelengths (given by $p = h/\lambda$) smaller than the nuclear size of a few fm to actually probe the nuclear structure; this requires $p \geq 10^{-19}$ kg m s^{-1}; for electrons the corresponding energy $E = (p^2c^2 + m^2c^4)^{1/2} \sim (10^{-21} + 10^{-26})^{1/2}$ J is about 200 MeV (requires an accelerator), while for alphas it is about 5 MeV (obtainable from natural radioactivity).

- Deducing nuclear size from scattering is not straight-forward. Experiments suggest a simple nuclear charge density distribution which is almost constant except for the surface region, described by the model
$$\rho_{\rm ch}(r) = \frac{\rho^0_{\rm ch}}{1 + e^{(r-R)/a}}$$
where $\rho^0_{\rm ch}$, R and a are adjustable parameters. It is found that $R \approx 1.12A^{1/3}$ fm, and $a \approx 0.52$ fm.

- The density $\rho^0_{\rm ch}$ probes the proton distribution. Assuming that neutrons are spread uniformly in the same volume, the nuclear density in most of the nucleus is about 0.17 nucleons fm^{-3}.

2 Binding energy

- The *binding energy* of a nucleus is a very important characteristic for stability, because of the various ways in which a nucleon can change to another or escape altogether. If the binding energy of the initial state is less than the total of the final states, *decay* (conversion of one system into a different one) is possible.

- The binding energy $B(Z, N)$ may be measured directly by disassembling a nucleus, but it is often convenient to measure it by measuring the mass of the nucleus, or of the (usually neutral) atom:
$$m_{\rm a}(Z, N) = Z(m_{\rm p} + m_{\rm e}) + Nm_{\rm n} - B(Z, N)/c^2 - b_{\rm e}/c^2$$
where $b_{\rm e}$ is the total electronic binding energy, ranging from 13.6 eV for H to several hundred keV for uranium.

- If we look at binding energy of light nuclei, we find (1) an average binding energy per nucleon that rises rapidly from about 1 MeV per particle for the deuteron to near 8 MeV per particle, (2) relatively high binding energy per nucleon for "even-even" nuclei such as ${}^{12}_{6}$C, and (3) rather small binding energy of the "last" nucleon for nuclei with one particle above even-even, rising to considerably larger values for the last nucleon as the next even-even nucleus is approached. There is clearly some important binding energy associated with neutron-neutron and proton-proton pairing.

- The binding energy per nucleon of ${}^4_2$He is so high that it is energetically advantageous for ${}^8_4$Be to break up into two He nuclei, in spite of its even-even nature. This is an example of the ease with which He nuclei (alpha particles) can be formed.

3 The semi-empirical mass formula

- The total binding energy of nuclei of various (Z, N) combinations may be fitted by a reasonably accurate approximate formula involving only a few parameters. One form of this equation is

$$B(Z,N) = aA - bA^{2/3} - s\frac{(N-Z)^2}{A} - \frac{dZ^2}{A^{1/3}} - \frac{\delta}{A^{1/2}},$$

where the parameters are determined by fitting the ensemble of measured binding energies. The best fits for this form are

$$\begin{aligned} a &= 15.835 \text{ MeV} \\ b &= 18.33 \text{ MeV} \\ s &= 23.20 \text{ MeV} \\ d &= 0.714 \text{ MeV}, \end{aligned}$$

and δ is 0 for even-odd nuclei, +11.2 MeV for odd-odd nuclei, and −11.2 MeV for even-even nuclei.

- In this formula, the various terms all have physical interpretations. The first term represents the mean energy per nucleon, approximately constant over the periodic table because the strong force, with its very short range, only binds nearest neighbors together. The second term is a surface term which reflects the decrease in binding energy due to nucleons on the surface where they are bound to fewer other nucleons than interior nucleons are. The fourth term describes the repulsive influence of the positive charges of the protons, and is approximately equal to the total energy of repulsion of a uniform sphere of total charge Ze within radius R.

- The third term reflects the effect of the exclusion principle, and is the simplest form that forces N to be approximately equal to Z. The reason for this "symmetry" term is that if we think of the energy levels within the nucleus as single-particle states like atomic orbitals, we can put two protons (spin up and spin down) into each level, and two neutrons. If we replace one of the protons by a neutron, the new neutron cannot go in the level of the former proton, but must go to the next level up. Thus it must have a higher energy than the proton it replaces. This effect favours $N = Z$. The A in the denominator makes this term proportional to the total number of nucleons for a fixed ratio N/Z.

- Finally, the term with δ describes the strong tendency of identical nucleons to prefer to occur in *pairs*. Thus even-even nuclei are on average considerably more strongly bound than odd-odd nuclei even if the last nuclear orbital is unfilled. The $1/A^{1/2}$ dependence is empirical; it reflects the fact that the pairing energy contribution is smaller in heavy nuclei than light ones.

4 The valley of β-stability

- Many nuclei are observed to be *unstable*: they spontaneously change into other nuclei by emission of one or more particles. The two common kinds of decay of naturally occurring nuclei are

  1. *beta-decay*: emission of an electron or positron, or absorption of an atomic electron, with conversion of a proton into a neutron or vice versa. An example is

$${}^{137}_{55}\mathrm{Cs} \rightarrow {}^{137}_{56}\mathrm{Ba} + \mathrm{e}^- + \bar{\nu}_\mathrm{e}.$$

2. *alpha-decay*: emission of a ${}^{4}_{2}$He nucleus (an alpha-particle). An example is

$$^{235}_{92}\mathrm{U} \to {}^{231}_{90}\mathrm{Th} + \alpha.$$

- Such decays occur when the nucleus in question can go to a state of lower energy by the decay, and enough energy is available from the transition to supply what is needed for the decay (e.g to create an electron and a neutrino).
- Consider beta-decay. In this decay, no nucleons are lost by the nucleus, so A stays constant, while $Z \to Z \pm 1$ and $N \to N \mp 1$.
- Although simple, the binding energy formula is accurate enough to predict correctly – in almost all cases – which nuclei are stable against beta-decay. Using our simple form, and substituting $A - Z$ for N, the mass of a particular atom is

$$\begin{aligned} m_{\mathrm{a}}(Z,N)c^2 &= [Nm_{\mathrm{n}} + Z(m_{\mathrm{p}} + m_{\mathrm{e}})]c^2 - \left[aA - bA^{2/3} - d\tfrac{Z^2}{A^{1/3}} + s\tfrac{(N-Z)^2}{A} + \delta\tfrac{1}{A^{1/2}}\right] \\ &= (Am_{\mathrm{n}}c^2 - aA + bA^{2/3} + sA + \delta A^{-1/2}) - (4s + (m_{\mathrm{n}} - m_{\mathrm{p}} - m_{\mathrm{e}})c^2)Z \\ &\quad + (4sA^{-1} + dA^{-1/3})Z^2. \end{aligned}$$

  This expression is a parabola in Z, opening upward.
- For odd A, $\delta = 0$ both before and after a beta-decay, since either Z or N is odd both before and after decay. In this case, we may find the most stable Z value for a given A by finding the minimum of the expression above:

$$Z = \frac{[4s + (m_{\mathrm{n}} - m_{\mathrm{p}} - m_{\mathrm{e}})c^2]A}{2(4s + dA^{2/3})}.$$

  The actual Z of lowest energy, Z_{min}, is the integer value nearest the computed Z. Since $dA^{2/3} = 0.714A^{2/3}$ MeV is larger than $(m_{\mathrm{n}} - m_{\mathrm{p}} - m_{\mathrm{e}})c^2 = 0.78$ MeV for $A \geq 2$ we see at once that $Z_{\mathrm{min}} \leq A/2$ and $N \geq Z_{\mathrm{min}}$.
- Since the ν is (nearly) massless, beta-decay is possible if

$$m_{\mathrm{nuc}}(Z,A) > m_{\mathrm{nuc}}(Z+1,A) + m_{\mathrm{e}}.$$

  Adding Zm_{e} to each side, this may be expressed using atomic masses as

$$m_{\mathrm{a}}(Z,A) > m_{\mathrm{a}}(Z+1,A).$$

- If $Z > Z_{\mathrm{min}}$, decay can occur by positron emission if $m_{\mathrm{nuc}}(Z,A) > m_{\mathrm{nuc}}(Z-1,A) + m_{\mathrm{e}}$ or $m_{\mathrm{a}}(Z,A) > m_{\mathrm{a}}(Z-1,A) + 2m_{\mathrm{e}}$.
- In an atom, another process that competes with positron emission is *electron capture* or *K-capture*. The nucleus absorbs an electron from its cloud (usually from the K shell), converting a proton to a neutron and emitting a neutrino. K-capture is possible if $m_{\mathrm{nuc}}(Z,A) + m_{\mathrm{e}} > m_{\mathrm{nuc}}(Z-1),A) + b_{\mathrm{e}}/c^2$, or $m_{\mathrm{a}}(Z,A) > m_{\mathrm{a}}(Z-1),A) + b_{\mathrm{e}}/c^2$. The binding energy b_{e} that must be supplied by the capture may be as much as 100 keV, but this is less than the two electron masses required by positron emission, so K-capture may be possible when β^+-emission is not.
- We thus expect that there will be only one stable isobar of a given odd A. This is in fact the case.
- Nuclei with even A either have both Z and N even, or both odd. Beta-decays will change an odd-odd nucleus to an even-even one, and vice versa. For these two situations, the binding energy term in δ has opposite signs, so there will be *two* mass parabolas for fixed A as a function of Z, one $2\delta/A^{1/2}$ above the other. Again β^- or β^+ decays are possible from higher to lower mass atoms, but now we generally expect to find one, two, or even three minimum Z values on the lower parabola, separated by higher odd-odd nuclei, and no minimum Z values on the upper parabola. This is observed to be true with two exceptions, ${}^{50}_{23}$V and ${}^{180}_{73}$Ta.

- An example (from third-year lab) of successive decays for even A leading to stability is

$$^{90}_{38}\text{Sr} \rightarrow {}^{90}_{39}\text{Y} + \beta^- + \bar{\nu}_\text{e} + 0.54\text{ MeV}$$

followed by

$$^{90}_{39}\text{Y} \rightarrow {}^{90}_{40}\text{Zr} + \beta^- + \bar{\nu}_\text{e} + 2.27\text{ MeV}.$$

The Zr is stable. Note that the second decay releases more energy than the first.

- To summarize, for given A (isobars) we expect to find only one stable Z if A is odd, and one, two or three if A is even. Furthermore, the trend of beta-stable nuclei in a diagram plotting N against Z (a "Chart of the Nuclides") is accurately reproduced.

5 Alpha-decay and fission

- The binding energy per nucleon, $B(Z, A)/A$, may be plotted as a function of A for the stable nuclei of Z_min. If the plot is restricted to odd-A nuclei, the scatter is quite small. $B(Z, A)/A$ rises from just over 1 MeV per nucleon in the deuteron to about 8.7 MeV per nucleon for the elements around iron ($Z \approx 26$, $A \approx 56$). From there it declines slowly to below 7.8 MeV per nucleon beyond $A \approx 200$.

- The positive binding energy is entirely contributed by the strong nuclear interaction, diminished somewhat by surface effects and by the effects of a difference between N and Z. However, the effect that actually causes $B(Z, A)/A$ to decline above $A \approx 60$ is the increasing Coulomb repulsion among the nucleons. Because the strong force acts only between nearest neighbors, it increases only as A (not A^2), while the electrostatic repulsion increases about as Z^2, and thus becomes more important as A and Z increase.

- The fact that the binding energy per nucleon B/A *decreases* beyond $A \approx 60$ means that decays in which the nucleus breaks into two smaller pieces may be able to release energy, and thus become possible.

- Of such decays, alpha-instability is the most common, because the decay leads to ejection of a tightly bound ^4_2He nucleus, making the 28.3 MeV binding energy of this nucleus available. All nuclei of $A > 165$ can release energy by ejecting an alpha-particle, but (as we will see later) the rate is often so slow as to be insignificant even over the age of the universe.

- Another possible decay mode is *fission*, in which a heavy nucleus breaks into two smaller, roughly equal fragments, each with higher B/A than the original nucleus.

6 The nuclear potential well

- The semi-empirical mass formula describes the stability behaviour of nuclei quite well. However, although we presented arguments for the form of each term, we fixed the coefficients purely empirically. We now look at simple quantum models of the nucleus and see what kinds of quantitative conclusions may be drawn about nuclear structure from such models.

- Consider what kind of potential we need to write down the Hamiltonian of the system. We do not know the details of the nuclear interaction. Let's suppose, as in many-electron atoms, that each individual nucleon moves in a spherically symmetric potential created by the other nucleons. This potential, because of electrostatic repulsion, will be somewhat different for protons than for neutrons. Since there is no attracting centre to the system (analogous to the atomic nucleus in a many-electron atom) we expect that the nuclear potential may be roughly describable as a strong force preventing each nucleon from leaving the nucleus, but the apparently uniform density within the nucleus suggests that there is no strong tendency for nucleons to be in any one part of the nucleus rather than another.

- This suggests that we could roughly describe the nuclear potential as a simple rectangular potential well with a flat floor. For neutrons, the well would have a depth and radius, and be zero outside the nucleus. For protons the well would be raised by electrostatic repulsion, and outside the nucleus would fall off to 0 at infinity from a substantial positive value. To really simplify matters, however, let's start by taking the wells to be infinitely deep.

- Now we estimate the parameters of these two wells. Note that nucleons are fermions, and thus obey the exclusion principle. Thus we expect that – roughly – they will fill up the lowest states in the nuclear potential, up to some upper level E_{F}. Each state will have one neutron and one proton of each spin direction. To estimate how many states are available, we need a *very* useful result concerning the density of states that we have met before.

- Consider the solution of the Schroedinger equation in a large box with sides at $x = 0$ and L, $y = 0$ and L, $z = 0$ and L. The potential is zero inside the box, and infinite outside. In the box,
$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) = E\psi(x,y,z).$$
We can clearly separate variables. Solutions satisfying $\psi = 0$ on the boundaries of the box are
$$\psi(x,y,z) = A\sin(k_x x)\sin(k_y y)\sin(k_z z)$$
where the wave numbers must satisfy
$$k_x = \pi n_x/L,$$
etc, with n_x, n_y and n_z *positive integers*, and (from the Schroedinger equation)
$$E = \hbar^2(k_x^2 + k_y^2 + k_z^2)/2m.$$

- The states with energies less than some particular energy (E_{max}, say) are all the values of k_x, etc for which $E \le E_{\mathrm{max}}$. Now the allowed states are *uniformly* spaced in **k**-space because of the relationship between k_x and n_x, etc, and in a small volume of **k**-space of $\Delta k_x \Delta k_y \Delta k_z$ there are $\Delta n_x \Delta n_y \Delta n_z = (L/\pi)^3 \Delta k_x \Delta k_y \Delta k_z$ states. Thus the total number of of allowed states in the box of energy up to E_{max} is
$$\mathcal{N}(E_{\mathrm{max}}) = (2)(\frac{1}{8})\frac{4\pi}{3}k_{\mathrm{max}}^3\left(\frac{L}{\pi}\right)^3 = \frac{V}{3\pi^2}\left(\frac{2mE_{\mathrm{max}}}{\hbar^2}\right)^{3/2},$$
where the factor 2 comes from two spin states, the factor 1/8 is the volume fraction in the quadrant having all positive values of k_x, etc (only positive values lead to distinguishable states), and V is the total volume of the box.

- Now apply this result to our nucleus. We have noted that the number densities of neutrons and protons are nearly equal and are constant independent of A, each at about 0.085 nucleons fm^{-3}. Equating this value to the $\mathcal{N}(\mathcal{E}_{\mathrm{max}})/V$ above, we deduce that the neutron (and proton) energy states in the nucleus will be filled up to a maximum (Fermi) energy E_{F} of about 38 MeV above the bottom of the nuclear potential well.

- The approximation above assumes an infinitely deep well, but does not change very much for a potential well of finite depth. We can estimate the total depth of the well by recalling that the typical binding energy per nucleon is about 8 MeV, so the total depth of the nuclear potential well is roughly 46 MeV.

- Using this simple model of single-nucleon states in the nuclear potential well, we can easily see what happens in nuclear beta-decay, when there is (say) an excess of protons over neutrons in the nucleus: a proton can release enough energy by becoming a neutron and dropping into a lower unoccupied neutron state to pay for the energy required to create the positron and neutrino.

7 Angular momentum of nuclear energy levels

- We may get more information about the nuclear energy levels in a nucleus by solving the Schroedinger equation for the well in three dimensions. For simplicity we assume again an infinitely deep well, so that the wave function of a proton or neutron is completely confined within a region of radius R around the origin of coordinates, and goes to zero on the boundary.

- Because we have assumed that the mean potential in which each nucleon moves is spherically symmetric, we know that the solution is separable. As before, we have $\psi(r, \theta, \phi) = u_l(r) Y_{lm}(\theta, \phi)$, with the spherical harmonics of definite angular momentum quantum numbers l and m_l describing the angular variation, and the radial wave function $u_l(r)$ satisfying

$$-\frac{\hbar^2}{2m_\mathrm{n}} \frac{1}{r} \frac{\mathrm{d}^2}{\mathrm{d}r^2}(r u_l) + \frac{\hbar}{2m_\mathrm{n}} \frac{l(l+1)}{r^2} u_l = E u_l.$$

  The boundary conditions are that $u_l(0)$ is finite and $u_l(R) = 0$.

- For $l = 0$ it is easy to verify that the solution is

$$u_\mathrm{s}(r) = \frac{\sin kr}{kr},$$

  with $E = \hbar^2 k^2 / 2m_\mathrm{n}$. To satisfy the boundary condition at R, k must satisfy $k = k_{n,\mathrm{s}} = n\pi/R$. This condition defines a series of $l = 0$ (s-state) energy levels of energies

$$E(n, \mathrm{s}) = \frac{\hbar^2}{2m_\mathrm{n}} \left(\frac{n\pi}{R^2}\right)^2.$$

- For $l = 1$ (p states), the Schroedinger equation becomes

$$-\frac{\hbar^2}{2m_\mathrm{n}} \frac{1}{r} \frac{\mathrm{d}^2}{\mathrm{d}r^2} r u_\mathrm{p} + \frac{\hbar}{2m_\mathrm{n}} \frac{1}{r^2} u_\mathrm{p} = E u_\mathrm{p},$$

  and the solution is

$$u_\mathrm{p}(r) = \frac{\sin kr}{(kr)^2} - \frac{\cos kr}{(kr)}.$$

  Again the boundary condition at $r = R$ requires $u_\mathrm{p}(R) = 0$. This condition is satisfied for a series of k values; the first few zeros of $u_\mathrm{p}(R)$ are at $k_{n,\mathrm{p}} R = 4.49,\ 7.73,\ 10.90 \ldots$. For each $k_{n,\mathrm{p}}$ we have a corresponding energy level $E(n, \mathrm{p})$, interleaved with the energy levels of the s-states.

- In general, the solutions of the Schroedinger equation for arbitrary l are the spherical Bessel functions

$$j_l(kr) = (-kr)^l \left(\frac{1}{k^2 r} \frac{\mathrm{d}}{\mathrm{d}r}\right)^l \left(\frac{\sin kr}{kr}\right).$$

  For each l, acceptable values of k are those which make $j_l(kR) = 0$. These zeros of the solution define a series of energy levels, interleaved with the levels of other values of l. In nuclear physics, we number these levels sequentially for each l, so the lowest few energy levels are 1s, 1p, 1d, 2s, 1f, 2p, 1g, 2d, 1h, 3s, etc. With degeneracy with respect to m_l and spin, there are $2(2l + 1)$ states of the same energy in each level (i.e. this number of protons and of neutrons may occupy "one" energy level).

- Thus, for this simple model, we find a series of energy levels, each of which can be occupied by only a limited number of protons and neutrons, and each of which has definite angular momentum. The situation is very reminiscent of atomic orbitals.

- This model can be improved in obvious ways, by making the well finite in depth rather than infinite, and by "softening" the outer boundary somewhat. It is found that these improvements lower the energy levels (relative to the ground state) somewhat, compared to the simple well with infinite walls, but do not change the order of single-nucleon levels much.

8 Magic numbers and spin-orbit coupling

- An important test of the shell model is the prediction of binding energy jumps analogous to those found in atoms for the filled shell configurations of the noble gases. For atoms, it is observed that the energy required to ionize a noble gas is three or four times larger than that required to ionize an atom which has one more electron than a noble gas (e.g. Ne and Na, Ar and K, Kr and Rb). For nuclei, a similar phenomenon is observed in several ways.

  1. Nuclei with certain values of Z (2, 8, 20, 28, 50, 82, 126) have unusually large numbers of stable isotopes.
  2. Similarly, unusually large numbers of isotones are found when N has these same values.
  3. Nuclei with Z or N near one of these numbers tend to be more spherical than usual (i.e. to have relatively small quadrupole moments).
  4. If we plot the energy required to separate the last neutron from a nucleus as a function of N and A, relatively large gaps are found at the same numbers mentioned above.

  These numbers are usually referred to as *magic numbers*.

- The simple shell model we have discussed predicts relatively large energy jumps after filling of the 1s, 1p, 2s, 1f, 2p, 1g, 1h, and 3p levels, which occur with 2, 8, 20, 34, 40 58, 92, and 138 neutrons or protons. The energy level spacing is thus *not* accurately predicted by this version of the shell model.

- The explanation was found independently by Maria Meyer and by O. Haxel et al in the late 1940's. They assumed that there is a strong coupling between $\mathbf{L}$ and $\mathbf{s}$ (note that generally there will be either zero or one unpaired spin, and $s = 1/2$), so that the potential seen by a nucleon contains not only the spherically symmetric term but also a term $U_{\rm so}(r)\mathbf{L}\cdot\mathbf{s}$. In this case, $\mathbf{L}^2$ and $\mathbf{s}^2$ are still good quantum numbers (conserved) because they commute with $\mathbf{L}\cdot\mathbf{s}$. Total angular momentum $\mathbf{J}$ is of course also conserved. However, since L_z and s_z do not commute with $\mathbf{L}\cdot\mathbf{s}$, they are no longer good quantum numbers. Nuclear states are then labeled with quantum numbers (l, s, j, j_z).

- The expectation value of $\mathbf{L}\cdot\mathbf{s} = (\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{s}^2)/2$ is $[j(j+1) - l(l+1) - s(s+1)]\hbar^2/2$, which has the value $l\hbar^2/2$ when $j = l + \frac{1}{2}$ ($\mathbf{L}$ and $\mathbf{s}$ are parallel) and $-(l+1)\hbar^2/2$ when $j = l - \frac{1}{2}$ ($\mathbf{L}$ and $\mathbf{s}$ anti-parallel). This interaction energy *splits* each level (n, l), which is $2(2l+1)$-times degenerate, into two separate levels that we label $nl_j = nl_{l+\frac{1}{2}}$ and $nl_{l-\frac{1}{2}}$.

- By taking the size of the spin-orbit splitting to be quite large (and opposite to that found in atomic electron clouds), it is possible to get the energy levels of the nuclear orbital shells to show splitting at the observed magic numbers.

- The shell model also predicts, for the most part correctly, the angular momentum of nuclear ground states. To predict these, note that we expect that all completely filled shells will contribute zero angular momentum (and positive parity). In partly filled shells, it appears that nucleons have a strong tendency to form *pairs* with opposite j_z values. Thus even numbers of nucleons in an unfilled shell still contribute zero angular momentum. The ground state angular momentum is determined by the remaining unpaired nucleon(s). For even-Z, even-N (even-even) nuclei, all nucleons will be paired, the total angular momentum will be zero, and the parity will be positive. For odd-even nuclei, the angular momentum j and parity of the ground state will be those of the single unpaired particle; the parity will thus be $(-1)^l$, and the angular momentum will be that of the shell currently being filled. For odd-odd nuclei, there is no systematic result.

9 Nuclear magnetic dipole moments

- In the shell model, we expect that even-even nuclei will have all orbital and spin angular momenta paired, so there will be no residual orbital or intrinsic spin currents, and no magnetic dipole moment. This prediction is in agreement with experiments.

- For odd-even nuclei, there is one unpaired nucleon. We expect that this nucleon will contribute a magnetic moment due to its orbital motion

  $$\mu_L = e\mathbf{L}/2m_{\mathrm{p}} = \mu_{\mathrm{N}}(\mathbf{L}/\hbar)$$

  for a proton, but 0 for a neutron. There will also be a contribution from the intrinsic spin

  $$\mu_{\mathrm{s}} = g_{\mathrm{s}}\mu_{\mathrm{N}}(\mathbf{s}/\hbar),$$

  where g_{s} is 5.59 for a proton and -3.83 for a neutron. Since neither $\mathbf{L}$ nor $\mathbf{s}$ has a defined z-component, the observed nuclear magnetic dipole moment should be the projection of the vector sum of the two individual magnetic moment components along the axis of the one vector defined for the nucleus, $\mathbf{J}$.

- To calculate the component of μ along $\mathbf{J}$, take the dot product of $\mathbf{J}$ with the operator

  $$\mu = \mu_L + \mu_{\mathrm{s}} = \mu_{\mathrm{N}}(g_L\mathbf{L} + g_{\mathrm{s}}\mathbf{s})/\hbar = \mu_{\mathrm{N}}[(g_L + g_{\mathrm{s}})(\mathbf{L}+\mathbf{s}) + (g_L - g_{\mathrm{s}})(\mathbf{L}-\mathbf{s})]/2\hbar,$$

  giving

  $$\mu \cdot \mathbf{J} = \mu_{\mathrm{N}}[(g_L + g_{\mathrm{s}})\mathbf{J}^2 + (g_L - g_{\mathrm{s}})(\mathbf{L}^2 - \mathbf{s}^2)]/2\hbar.$$

  Taking the expectation value of both sides of this expression, we find

  $$\mu = (\mu_{\mathrm{N}}/2)\left[(g_L + g_{\mathrm{s}})j + (g_L - g_{\mathrm{s}})\frac{(l-s)(l+s+1)}{(j+1)}\right].$$

  Now recall that $s = \frac{1}{2}$ and $j = l \pm \frac{1}{2}$, and find the expected magnetic moment from the unpaired nucleon to be

  $$\mu = \mu_{\mathrm{N}}\left[jg_L - \frac{1}{2}(g_L - g_{\mathrm{s}})\right]$$

  for $j = l + \frac{1}{2}$ and

  $$\mu = \mu_{\mathrm{N}}\left[jg_L - \frac{j}{2(j+1)}(g_L - g_{\mathrm{s}})\right]$$

  for $j = l - \frac{1}{2}$.

- These two values are called the “Schmidt lines” in a plot of μ versus j. It is found that the observed magnetic moments mostly fall between these two values, not on them. This may be due to mixing of shell model states, or to a change in the intrinsic magnetic moments of the proton and neutron in the nuclear environment.