Beta and gamma decays

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1 Simple Fermi theory of beta decay

- Beta decay is one of the most easily found kinds of radioactivity. As we have seen, this result of the weak interaction leads to conversion of a neutron into a proton or vice versa, with the necessary charge change being made possible by the emission of a positive or negative beta particle (positron or electron). To ensure conservation of lepton number, each such event is accompanied by emission of an electron neutrino. Alternatively, an orbital electron may be absorbed (electron capture or K-capture), changing a proton into a neutron (this process competes with β^+ emission), with emission of an electron neutrino.

- The spectra of electrons or positrons emitted in beta decay is a continuum of energies, up to a maximum value, with most emitted betas having intermediate energies. The emitted, unobserved, neutrinos also have a continuous energy distribution. In K-capture, there is no electron emitted; the neutrino is mono-energetic. (Mono-energetic electrons also emerge from de-excitation of an excited nucleus by internal conversion [see below]; this process leads to electron emission but is quite a different physical mechanism than beta decay.)

- We can predict the general form of the energy spectrum of the (observable) beta particle, but not the absolute decay rate, from a simple theory proposed by Fermi. The form of the spectrum is found simply by considering the density of final states!

- We have seen, using perturbation theory, that the probability per unit time of a transition into a group of closely spaced states is given by

$$W_{\rm fi} = \frac{2\pi}{\hbar}|H_{\rm fi}|^2 n_{\rm f}(E_{\rm f}),$$

  where $n_{\rm f}(E_{\rm f})$ is the density of final states and $H_{\rm fi}$ is the matrix element of the Hamiltonian between states i and f (Lecture 2).

- In beta decay of an even-odd nucleus, the initial state is simply the wave function of the single odd nucleon (let's say it is a proton), $\Psi_{\rm i} = \psi_{\rm p}(\mathbf{r}_{\rm p})$. The final state

is the wave function of the new neutron, the emitted positron and the neutrino: $\Psi_{\rm f} = \psi_{\rm n}(\mathbf{r}_{\rm n})\psi_{\rm e}(\mathbf{r}_{\rm e})\psi_\nu(\mathbf{r}_\nu)$. The interaction Hamiltonian is uncertain, but very short range, and Fermi tried the simplest operator he could think of, namely a constant that we will call $G_{\rm w}$. Thus the matrix element is only non-zero to the extent that the initial and final wave functions overlap:

$$H_{\rm fi} \approx G_{\rm w} \int \psi_{\rm n}^*(\mathbf{r})\psi_{\rm e}^*(\mathbf{r})\psi_\nu^*(\mathbf{r})\psi_{\rm p}(\mathbf{r}){\rm d}\mathbf{r}^3.$$

- For the lepton wave functions we take plane waves (although this is not accurate for the beta particle, which is affected by the Coulomb interaction with the nuclear protons). These must be normalized, so we again introduce an artificial (large) volume V:

$$\psi_\nu^*(\mathbf{r}_\nu) = \frac{1}{V^{1/2}} e^{i\mathbf{k}_\nu\cdot\mathbf{r}_\nu}$$

and

$$\psi_{\rm e}^*(\mathbf{r}_{\rm e}) = \frac{1}{V^{1/2}} e^{i\mathbf{k}_{\rm e}\cdot\mathbf{r}_{\rm e}}$$

- Now for lepton energies of a few MeV or less, the wavelengths of the lepton wave function are of order 1 MeV/$\hbar c \sim 10^{-2}$ fm^{-1}. Thus the exponent of each of the exponentials is small over the extent of the nucleus, and we may replace the two lepton wave functions by $V^{-1/2}$. Then our trial matrix element is just

$$H_{\rm fi} \approx \frac{G_{\rm w}}{V} \int \psi_{\rm n}^*(\mathbf{r})\psi_{\rm p}(\mathbf{r}){\rm d}^3\mathbf{r}.$$

In general we do not know what the value of the integral is. Let us simply assume that it does not depend on how the released energy is partitioned between the outgoing beta particle and the neutrino, so that for a given decay we may treat it as a constant which we will call $M_{\rm F}$.

- We have so far ignored the (significant) effect of the Coulomb interaction between the outgoing beta particle and the nucleus. The effect of this interaction for non-relativistic particles is to replace the beta particle plane wave at the origin by the wavefunction of the beta particle in the electric field of the daughter nucleus of charge $Z_{\rm d}$. This is equivalent to multiplying our expression for $W_{\rm fi}$ by $F(Z, E_\beta) = |\psi_\beta(Z_{\rm d}, 0)/\psi_\beta(0, 0)|^2$. It is found that

$$F(Z, E_\beta) \approx \frac{2\pi\eta}{1 - \exp(-2\pi\eta)}$$

where $\eta = \pm(Ze^2)/(4\pi\varepsilon_0\hbar v_\beta)$ and the sign of η is opposite to that of the charge on the beta particle. For both electrons and positrons $F(Z, E_\beta) > 0$; for large beta velocity this factor approaches 1 (as it should) but for small v_β the function tends to a large value for electrons (for which the outgoing wavefunction is concentrated around the nucleus) and to a small value for positrons (for which the outgoing wavefunction is made small near the nucleus by the Coulomb repulsion).

- So finally the decay rate that we need is

$$W_{\rm fi} \approx \frac{2\pi}{\hbar}\frac{G_{\rm w}^2}{V^2}|M_{\rm F}|^2 F(Z,E_\beta) n_{\rm f}(E_{\rm f}).$$

- What density of final states is needed? After the decay, we may suppose that the final nuclear state is stable, so it has a well-defined energy, while the initial unstable state has finite width due to the uncertainty principle $\Delta E_{\rm i}\Delta t \geq \hbar$. Suppose that in the decay we can measure precisely the state of the outgoing electron (its momentum vector, not its position, of course). Now because of the uncertainty in initial energy, the outgoing neutrino with have a slightly uncertain energy, and may end up in any one of a number of closely adjacent states. Thus for the probability of decay to a *specific* final electron state, we need the density of final *neutrino* states.

- Now recalling that the neutrino is relativistic (and may have non-zero mass), so that its momentum and energy are related by $E_\nu^2 = (m_\nu c^2)^2 + (p_\nu c)^2$, we may use our earlier result that the density of plane wave states (neglecting spin) in V is $n_{\rm f} = [V/(2\pi)^3]4\pi k_\nu^2 {\rm d}k_\nu$, where $k_\nu = (E_\nu^2 - m_\nu^2 c^4)^{1/2}/\hbar c$ and $E_\nu {\rm d}E_\nu = \hbar^2 c^2 k_\nu {\rm d}k_\nu$. Thus the relativistic density of neutrino states is

$$n_\nu(E_\nu){\rm d}E_\nu = \frac{V}{(2\pi)^3}\frac{4\pi}{(\hbar c)^3}(E_\nu^2 - m_\nu^2 c^4)^{1/2} E_\nu {\rm d}E_\nu.$$

- However, because the outgoing neutrino is not observable, we need to express this in terms of the observable beta particle energy, $E_\beta = E_0 - E_\nu$, where E_0 is the total energy available for the decay. Thus the probability of decay into a *single* electron state of energy E_β is

$$W_{\rm fi} \approx \frac{G_{\rm w}^2 |M_{\rm F}|^2}{\pi V \hbar^4 c^3} F(Z,E_\beta)(E_0 - E_\beta)[(E_0 - E_\beta)^2 - (m_\nu c^2)^2]^{1/2},$$

and the total probability of decay into *any* of the final beta particle states within momentum interval ${\rm d}p_\beta$ – using now the density of *beta* particle states – is

$$\begin{aligned} P(p_\beta){\rm d}p_\beta &\approx W_{\rm fi} n(p_\beta){\rm d}p_\beta \\ &\approx \frac{G_{\rm w}^2 |M_{\rm if}|^2}{2\pi^3 \hbar^7 c^3} F(Z,E_\beta) p_\beta^2 (E_0 - E_\beta)[(E_0 - E_\beta)^2 - (m_\nu c^2)^2]^{1/2} {\rm d}p_\beta. \end{aligned}$$

The distribution of electron energies (or momenta) is produced *entirely* by the density of states factors.

- This theory of beta decay is usually tested, and the value of the total decay energy determined accurately, by plotting the value of $\sqrt{P(p_\beta)/p_\beta^2}$ against the total or kinetic energy. Such a plot is called a *Kurie plot*; it has the virtue that if m_ν is small, the measured points are proportional to $E_0 - E_\beta$ and the graph crosses 0 where $E_0 = E_\beta$. From such graphs it is found that the rest mass-energy of the neutrino is no more than about 5 eV.

- The total transition rate for a decay is found by integrating the expression above over beta particle energy. The resulting mean lifetime of a beta-unstable nucleus is

$$\frac{1}{\tau} \approx \frac{G_{\rm wk}^2 |M_{\rm F}|^2 m_{\rm e}^5 c^4}{2\pi^3 \hbar^7} f(Z_{\rm d}, E_0),$$

where

$$f(Z_{\rm d}, E_0) = \left(\frac{1}{m_{\rm e}c^2}\right)^5 \int_{m_{\rm e}c^2}^{E_0} F(Z, E_\beta)(E_0 - E_\beta)^2 (E_\beta^2 - m_{\rm e}^2 c^4)^{1/2} E_\beta {\rm d}E_\beta,$$

neglecting the neutrino mass. The function $f(Z_{\rm d}, E_0)$ is dimensionless and has been extensively tabulated. Once the value of the constant $G_{\rm w}$ has been determined (from an experiment in which the matrix element can be evaluated approximately), other experiments determine experimentally the value of the unknown matrix elements of the decay in terms of the quantity $f(Z_{\rm d}, E_0)t_{1/2}$. This *ft-value* is the quantity usually quoted for an experimentally studied beta decay, rather than the value of the matrix element.

1.1 Electron capture

- A process that competes with positron beta decay is electron capture, in which the same conversion of a proton into a neutron is accomplished by the capture of an *orbital* electron with emission of a neutrino. We may calculate the rate of this process using very similar reasoning to that discussed above.

- Again we use Fermi's golden rule. Now the initial state is a nuclear proton wave function and an orbital electron wave function, while the final state is an nuclear neutron and a free neutrino of (almost) definite energy (the initial state, of finite lifetime, has a slightly uncertain energy). The initial electron state (assuming the electron to be a K electron in the innermost shell) is thus

$$\psi_{\rm e}(\mathbf{r}) = \frac{1}{\sqrt{\pi}} \left(\frac{Z m_{\rm e} e^2}{4\pi\varepsilon_0 \hbar^2}\right)^{3/2} \exp\left(-\frac{Z m_{\rm e} e^2 r}{4\pi\varepsilon_0 \hbar^2}\right).$$

Thus the total rate for this process is

$$\frac{1}{\tau_{\rm K}} \approx 2\frac{2\pi}{\hbar} |H_{\rm fi}|^2 n_\nu(E_\nu),$$

where the initial 2 arises because there are two K-shell electrons, the matrix element is given by

$$H_{\rm fi} = G_{\rm w} \int \psi_{\rm n}^*(\mathbf{r}) \psi_\nu^*(\mathbf{r}) \psi_{\rm p}(\mathbf{r}) \psi_{\rm e}(\mathbf{r}) {\rm d}^3\mathbf{r},$$

and the density of states is only that of the neutrino,

$$n_\nu(E_\nu) = \frac{V}{(2\pi)^3} \frac{4\pi}{\hbar^3 c^3} E_\nu^2.$$

Because the electron wave function is normalized to its confined volume around the nucleus, V only appears to the first power now. Again we simply need to evaluate the electron wave function at the nucleus (i.e. at $r = 0$), and the only part of the matrix element that we cannot evaluate is the nuclear overlap integral, which is the same integral $M_{\rm F}$ that appears in the normal beta decay theory.

- Putting the pieces together, the transition rate for electron capture (the inverse of the lifetime) is

$$\frac{1}{\tau_{\rm K}} \approx \frac{2G_{\rm w}^2 |M_{\rm F}|^2 E_{0{\rm c}}^2}{\pi^2 \hbar^4 c^3} \left(\frac{Z m_{\rm e} e^2}{4\pi\varepsilon_0 \hbar^2} \right)^3 ,$$

  where $E_{0{\rm c}}^2$ is the energy available in the decay (given to the escaping neutrino). Note that this result is two times larger than in C & G because they have calculated the rate for a single K electron.

- The ratio of the decay rate by this process to normal positron decay is easily calculated because the same nuclear matrix element appears in both results. We get

$$\frac{1/\tau_{\rm K}}{1/\tau} = 4\pi \left(\frac{E_{0{\rm c}}}{m_{\rm e} c^2} \right)^2 \frac{(\alpha Z)^3}{f(Z_{\rm d}, E_{0{\rm c}})},$$

  where $\alpha = (e^2/(4\pi\varepsilon_0 \hbar c) \approx 1/137$ is the (dimensionless) fine structure constant (Lecture 3) and we neglect the mass of the neutrino. We can see that for small available energies this ratio is made large by the smallness of $f(Z_{\rm d}, E_{0{\rm c}})$ (and is infinite if positron decay is not permitted energetically); it is also relatively large for high Z because of the Z^3 factor.

- Experiments on electron capture are not as straight-forward as on normal beta decay. The outgoing neutrino is unobservable. Instead, what may be detected is an x-ray from transitions in the electron cloud as the missing K electron is replaced from a higher level, and/or an Auger electron ejected from the excited electron cloud, carrying off the energy released as an L-shell electron drops into the K-shell.

2 Gamma decays

- You have already gone through the (partial) derivation of the spontaneous emission rate of photons from an excited system (Secs 4.2 and 4.3 in B & J). The final result of this exercise was to derive the lowest order emission probability per second, or inverse lifetime, in the dipole approximation,

$$W_{ab}^s = \frac{4\alpha}{3c^2} \omega_{ba}^3 |\mathbf{r}_{ba}|^2,$$

  where ω_{ba} is the angular frequency corresponding to the energy difference between states a and b, and $\mathbf{r}_{ba} = \langle \psi_a | \mathbf{r} | \psi_b \rangle$ is the matrix element of the position vector between the wavefunctions of states a and b.

- The derivation of this result (repeated in somewhat different form in C & G) is as valid for nuclear photon emission as for emission from atoms: B & J Eq. [4.71] is identical to C & G (12.12). The same approximations are valid, particularly the expansion of the outgoing wave state $\exp(i\mathbf{k}\cdot\mathbf{r})$ in powers of $\mathbf{k}\cdot\mathbf{r}$. Clearly since nuclei are much smaller than atoms, the spatial extent of the wavefunction will be much smaller, resulting in a much smaller value for $|\mathbf{r}_{ba}|$, but this is compensated for by much larger values of ω_{ba} than are found for atoms. In fact the order of magnitude of the emission rate for allowed nuclear transitions,

$$\frac{1}{\tau_{\mathrm{E1}}} \sim 0.4 \times 10^{15} \left(\frac{E_\gamma}{1\ \mathrm{MeV}}\right)^3 \left(\frac{|\mathbf{r}_{\mathrm{fi}}|}{1\mathrm{fm}}\right)^2 \ \mathrm{s}^{-1}$$

  is quite a lot *larger* than the characteristic rate for allowed atomic transitions

$$\frac{1}{\tau_{\mathrm{E1}}} \sim 0.4 \times 10^{7} \left(\frac{E_\gamma}{1\ \mathrm{eV}}\right)^3 \left(\frac{|\mathbf{r}_{\mathrm{fi}}|}{1\ \mathrm{\AA}}\right)^2 \ \mathrm{s}^{-1}.$$

- Since atoms – in most interesting contexts – collide with one another and so can de-excite collisionally if radiation is forbidden (i.e. highly suppressed), they rarely emit much radiation which is not allowed in the lowest dipole approximation. In contrast, nuclei often find themselves with no means of de-exciting from an excited state other than photon radiation. If an excited state differs from all lower states by several units of angular momentum, radiation will be possible only by a rather high multipole, and since each higher multipole of electric radiation is slower at nuclear energies by a factor of order 10^4 than the next lower multipole, lifetimes of radiative decay can sometimes be quite long even by human standards. In this case we speak not of metastable but of *isomeric* states; such states may be stable enough to make it into the handbooks....

- Another (electromagnetic) process which competes with radiation, especially if it is rather forbidden, is *internal conversion*. In this process, the nucleus transfers its excitation energy directly to an atomic electron (usually a K-shell electron) which is ejected, carrying off the excitation energy. The matrix elements of the two processes are similar, but the increasing concentration of the K-shell electrons near the nucleus as Z increases means that internal conversion competes increasingly well with photon emission as Z increases. Note that there is no photon involved here; the energy is transferred *directly* to the atomic electron.