

Nuclear fusion in stars and laboratories

April 1, 2002

1 Stellar structure and stellar energy

1.1 Stellar structure and evolution

- We observe stars in the sky. They are (to order of magnitude) objects like our sun, with characteristic mass of order $M \sim 10^{30}$ kg, characteristic size of $R \sim 10^9$ m (well, with a range of 10^4 or so either way), and characteristic luminosity of $L \sim 10^{26}$ W (again with a large range).
- Stars are observed to be *constant* (nearly, anyway). Since the *dynamical time scale* is of the order of the free-fall time,

$$\tau_{\text{ff}} \sim \left(\frac{R}{g}\right)^{1/2} \sim \left(\frac{R^3}{GM}\right)^{1/2} \sim 1 \text{ hour},$$

stars are almost always in dynamical equilibrium.

- Stars are supported against gravity (until they grow old) by gas pressure. From the equation of hydrostatic equilibrium,

$$\frac{dP}{dr} = -\rho g$$

where P is pressure, r is the radial coordinate from the centre of the spherical star, and $g = GM_r/r^2$ is the local gravity due to the enclosed mass M_r at r , we may estimate the order of magnitude of the central pressure. Replace the derivative $dP/dr \rightarrow [P(R) - P(0)]/R = -P(0)/R$ and estimate $\rho g \rightarrow (M/R^3)(GM/R^2) = GM^2/R^5$. Solve for $P(0)$ which is of order $P(0) \sim 10^{14}$ Pa. Similarly we estimate the mean density to be of order $\bar{\rho} \sim M/R^3 \sim 10^3$ kg m⁻³ (about the same as condensed matter).

- The support pressure in visible stars, in spite of their rather high density, is gas pressure (the atoms are highly ionized, and so they are considerably smaller than a_0). We may estimate the temperature using the ideal gas law and the fact that – from spectroscopy – we find that stars are overwhelmingly composed of H (i.e. $A \approx 1$): using $\bar{p} \sim \mu m_{\text{u}} n_i$ where n_i is the number density of ions and

$\mu \approx 1$ for H, we find $n_i \sim 10^{30}$ ions m^{-3} . From $P = (n_i + n_e)k_B T$, and the fact that each ionized H contributes one electron to n_e , we find $T \sim 3 \times 10^6$ K as the characteristic interior temperature for a star. All of these estimates are confirmed by detailed modelling.

- With such a high internal temperature, a star leaks heat (for the sun at the rate of $L_\odot = 3.9 \times 10^{26}$ W). Without any source of energy other than that provided by gravitational contraction, it must change on a thermal time scale of

$$\tau_{\text{th}} \approx \frac{\text{internal energy}}{\text{luminosity}} \approx \frac{3}{2} \frac{M}{\mu m_{\text{u}}} \frac{1}{L},$$

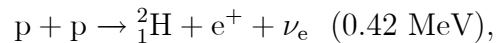
which for the sun is of the order of 10^7 yr. Geologists tell us that the climate on the earth has changed little over more than 3×10^9 yr, and so there must be some additional source of energy. By the late 1930's it had become clear that this must be a nuclear energy source.

- Nuclear energy is quite capable in the case of the sun of providing the entire energy output. From the typical binding energy of a nucleon in a nucleus of 8 MeV, compared to the value of $mc^2 \approx 940$ MeV, we see that combining H into heavier nuclei releases about 1% of the mass energy as other forms of energy. Converting 1% of the sun's mass into energy could release about 2×10^{45} J, enough to supply the current luminosity for 10^{11} yr.
- We have thus been led to the following view of stellar evolution.
 1. Stars form by collapse of interstellar gas clouds under the influence of gravity. As they shrink, they release gravitational energy. Once a protostar has become sufficiently dense to be opaque, this energy cannot be freely radiated away, and the internal temperature starts to rise. This provides pressure support, so the protostar shrinks on a thermal time scale.
 2. As the star shrinks, the internal temperature rises. Eventually it reaches a value of the order of 10^7 K, high enough to begin fusing H nuclei into He (we will look at the nuclear details soon). Energy release from this reaction replaces energy that leaks out of the star, allowing the star to enter a long period of nuclear equilibrium known as the *main sequence phase*.
 3. Eventually the H fuel near the centre (where conditions are right for nuclear reactions) is exhausted. The star starts to evolve on a thermal time scale again, with the under-supported core shrinking (and a low density envelope expanding to a large size). The temperature at the centre continues to rise, making possible other nuclear reactions formerly inhibited by the Coulomb barrier, such as the conversion of He into C and O. These reactions sustain the star for a while.
 4. Finally, all useful sources of nuclear energy are exhausted. The star shrinks or collapses until it finds a structure in which its mass can be supported by electron degeneracy (a white dwarf) or nuclear degeneracy (a neutron

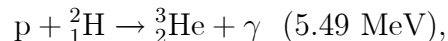
star), or it collapses completely as a black hole. If it becomes a neutron star or black hole, the collapse is accompanied by an enormous gravitational energy release, and the star briefly becomes a supernova.

1.2 Stellar nuclear reactions

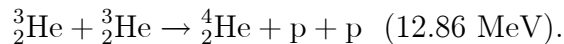
- We now look at the fusion reactions that power the main sequence life of a star. Since a star is (initially) mainly composed of H, we look for reactions using this nucleus, which has the lowest Coulomb barrier. We need a reaction that can go in spite of temperatures corresponding to particle energies of only $k_B T \sim 1$ keV.
- The big bottleneck to nuclear reactions involving H – apart from the Coulomb barrier – is the fact that the product of the collision of two protons, ${}^2\text{He}$, is *completely* unstable on a nuclear time scale of 10^{-23} s. The only way around this problem is for one of the two protons, while they are briefly in contact, to undergo beta decay and produce a deuteron!
- The initial reaction which makes stellar fusion possible is thus



often written in obvious notation as ${}^1_1\text{H}(p, e^+){}^2_1\text{H}$. This reaction is mostly followed by conversion of deuterium into helium-3,

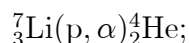
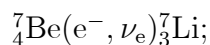
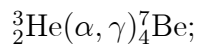


and when the abundance of ${}^3_2\text{He}$ has increased to a significant level, a final reaction leads to the formation of ${}^4_2\text{He}$ through

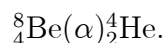
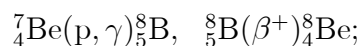


This cycle (the “PPI chain”) results in the release of 6.55 MeV per proton consumed, or 26.2 MeV per He nucleus formed, plus about 0.5 MeV that escapes the star directly in the form of neutrinos.

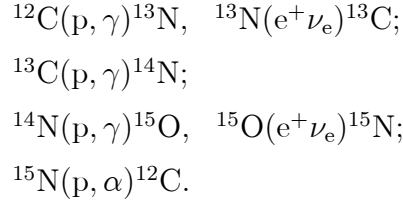
- This is not the only pathway followed by nuclear reactions in the centre of the sun; as is usually the case inside stars, a number of reactions take place more or less simultaneously. One side branch of the chain above is the “PPII chain”



and in turn a branch from this cycle is the “PPIII chain”



- To a small extent in the sun, and to a much larger extent in hotter stars where temperatures and particle energies are hotter, another cycle catalyzed by C, N and O (called the “CNO cycle”) leads to the same basic result, conversion of H into He with release of energy



This reaction is able to compete with the PP chains – in spite of a considerably higher Coulomb barrier – because the weak decays occur in nuclei that do not fall apart while waiting for beta-decay to happen. Note also that this reaction cycle requires the prior presence of C (or N or O) nuclei (in the sun, about 1 nucleus in 2500 is C), and does not lead to permanent conversion of C to heavier nuclei – at the end of the cycle, the original ^{12}C nucleus is replaced.

1.3 Stellar nuclear reaction rates

- Stellar nuclear reactions begin at temperatures that correspond to particle energies very low compared to the Coulomb barriers; even under these unfavourable conditions enough energy can be released to replace that leaked out into space. Thus we are in a situation where most of the reactions will be due to particles in the high energy tail of the thermal energy distribution, and even for these particles there is a major tunneling barrier. We now consider how the reaction rates needed for theoretical stellar models can be calculated.
- We have seen that the reaction rate per target nucleus a is $n_b v \sigma$, where n_b is the number density of bombarding particles b , v is the relative velocity, and σ_{ab} is the total reaction cross section, for example for the first step of the PPI chain. Because the probability of interaction depends strongly on particle velocity, we will have to average this rate over the velocity distribution. Thus the reaction rate R per unit volume of gas is

$$R = K n_a n_b \langle v \sigma_{ab} \rangle,$$

where

$$\langle v \sigma_{ab} \rangle = \int_0^\infty v \sigma_{ab} P(v) dv,$$

and the factor K is 1 for reactions between dissimilar particles (for example protons and C nuclei), and 1/2 for reactions between identical particles (this 1/2 prevent counting the same reaction twice).

- For the cross section we will use the low-energy limiting form derived previously,

$$\sigma(E) = \frac{1}{E} S(0) e^{-\sqrt{E_G/E}},$$

where $E_G = 2mc^2[\pi Z_1 Z_2 e^2 / (4\pi\epsilon_0)\hbar c]$ in general; for proton-proton reactions both Z 's are 1.

- The probability of two nuclei having a relative velocity v , or a relative kinetic energy E , is given by the Maxwell-Boltzmann distribution

$$P(v)dv = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{k_B T}\right)^{3/2} e^{-mv^2/2k_B T} v^2 dv$$

or

$$P(E)dE = \left(\frac{2}{\sqrt{\pi}}\right) \left(\frac{1}{k_B T}\right)^{3/2} e^{-E/k_B T} E^{1/2} dE,$$

where m is the reduced mass of the interacting pair. Using the low energy form of the reaction cross section, and changing to $E = mv^2/2$ as the variable, we have

$$\langle v\sigma_{pp} \rangle = \left(\frac{8}{\pi m}\right)^{1/2} \left(\frac{1}{k_B T}\right)^{3/2} S_{pp}(0) \int_0^\infty e^{-\phi(E)} dE,$$

where $\phi(E) = E/k_B T + \sqrt{E_G/E}$.

- This integral cannot be done analytically (although there is no problem in doing it numerically). But the integrand is strongly peaked around the value $E_0 = E_G^{1/3}(k_B T)^{2/3}$ where $\phi(E)$ is a minimum; for somewhat smaller E values the integrand drops quickly to zero because of the Coulomb barrier, while for larger E it goes to zero because of the decrease of the Boltzmann factor. We may evaluate the integral approximately by using a parabolic Taylor expansion of $\phi(E)$ around E_0 ,

$$\phi(E) \approx \phi(E_0) + \frac{1}{2}(E - E_0)^2 \phi''(E_0),$$

where

$$\begin{aligned} \phi(E_0) &= \frac{3}{2^{2/3}} (E_G/k_B T)^{1/3} \\ \phi''(E_0) &= \frac{3}{2^{2/3}} E_G^{-1/3} (k_B T)^{-5/3}. \end{aligned}$$

The term with $\phi'(E_0)$ is zero because we are expanding around E_0 . In this approximation, the integrand is a Gaussian, and the integration limits may be taken as $\pm\infty$. Using $\int_{-\infty}^\infty e^{-ax^2} dx = (\pi/a)^{1/2}$, we find

$$\langle v\sigma_{pp} \rangle = \frac{8}{9} S_{ab}(0) \left(\frac{2}{3mE_G}\right)^{1/2} \tau^2 e^{-\tau},$$

with

$$\tau = \frac{3}{2^{2/3}} \left(\frac{E_G}{k_B T}\right)^{1/3} = 3 \left(\frac{mc^2}{2k_B T}\right)^{1/3} \left(\frac{\pi Z_a Z_b e^2}{4\pi\epsilon_0 \hbar c}\right)^{2/3}.$$

Practical expressions are given by C & G. Notice that the only dependence on T is through the factor $\tau^2 \exp(-\tau)$. Typically τ is of the order of 20 for

p-p reactions at 10^7 K, so the exponential factor is *very* small; nevertheless the number of collisions is large enough for this reaction to make possible nuclear release of 10^{26} W.

- For your interest, the factors $S(0)$ for a few reactions (for energies well below important resonances) are listed in the table below (these are rather old values, taken from Fowler et al 1967, *Ann Rev Astr Ap* 5, 528, which however provides an excellent summary of the use of nuclear data in stellar computations). You

Reaction	$S(0)$	Energy release Q , MeV
${}^1\text{H}(p, \beta^+){}^2\text{H}$	3.4×10^{-25}	0.42
${}^2\text{H}(p, \gamma){}^3\text{He}$	2.5×10^{-7}	5.49
${}^3\text{He}({}^3\text{He}, 2p){}^4\text{He}$	5.00	12.86
${}^3_2\text{He}(\alpha, \gamma){}^7_4\text{Be}$	4.70×10^{-4}	1.59
${}^7_3\text{Li}(p, \alpha){}^4_2\text{He}$	1.25×10^{-1}	17.35
${}^7_4\text{Be}(p, \gamma){}^8_5\text{B}$	4.00×10^{-5}	0.14
${}^{12}\text{C}(p, \gamma){}^{13}\text{N}$	1.4×10^{-3}	1.94
${}^{13}\text{C}(p, \gamma){}^{14}\text{N}$	5.5×10^{-3}	7.55
${}^{14}\text{N}(p, \gamma){}^{15}\text{O}$	2.75×10^{-3}	7.29
${}^{15}\text{N}(p, \alpha){}^{12}\text{C}$	5.34×10^1	4.97

can see that $S(0)$ is of the order of 1 for reactions which involve only the strong interaction, of order 10^{-3} or less for reactions in which a γ must be emitted, and of order 10^{-25} if a weak decay must occur to allow the reaction to proceed. This is a measure of the weakness of the weak interaction.

1.4 Solar neutrinos

- The sun produces enormous numbers of neutrinos (of order $10^{15} \text{ m}^{-2} \text{ s}^{-1}$) mostly from the PPI reactions but also from all the alternative chains (since it is always necessary to convert two p's to n's to make each ${}^4\text{He}$ nucleus). The PPI neutrinos have energies of 0.42 MeV or less, but a tiny fraction of the others have energies above 10 MeV.
- There is much interest in observing solar neutrinos both to test our models of the sun and to study neutrino physics. Three basic methods are used.
 1. Low energy *electron* neutrinos may be studied using neutrino capture by neutrons in the elements as ${}^{71}_{31}\text{Ga}$ (which changes into Ge) in the GALLEX experiment in Italy, or ${}^{37}_{17}\text{Cl}$ (which becomes Ar) in the Homestake Mine

in the US. The beta-unstable nuclei produced are detected using x-rays emitted following K-capture.

2. The very high energy neutrinos (above about 5 MeV) are studied by observing them scattering from electrons in *very* pure water. Neutrinos entering water are virtually unable to interact with the nuclei – even an electron neutrino cannot react with a single proton in the H, and the threshold for reaction in ^{16}O is almost 15 MeV, too high to capture more than a tiny number of neutrinos. Thus elastic scattering off electrons is all that is left. This scattering is produced by *all* flavours of neutrinos, and is observed via Čerenkov radiation detected by large photomultipliers. Some 20 kilotonnes of water is used in the Super Kamiokande facility in Japan (which recently suffered catastrophic failure).
 3. The Sudbury Neutrino Observatory (SNO) is a kind of hybrid of the two methods above. It has a detector with one kilotonne of heavy water (D_2O) which up to now has been used to observe Čerenkov radiation from energetic electrons produced by $^2\text{H}(\nu_e, e^-)\text{p}^1\text{H}$ (conversion of the neutron in deuterium into a proton). Up to now the threshold for detection has been similar to that in Japan.
- The cross section for (electron) neutrino capture by a neutron is a lot higher than for scattering by electrons, so neutrino capture experiments don't need as much target material as a scattering experiment (30 tonnes at GALLEX, 600 tonnes in the Homestake Mine, 1 kilotonne in Sudbury, compared to 20 kilotonnes at Super Kamiokande).
 - The detection rate of high energy neutrinos by SNO (which detects only ν_e 's) is lower than that observed at Super Kamiokande (which detects all flavours). It appears that this provides strong direct evidence that neutrinos oscillate between their flavours as they travel from the sun to the earth.

2 Fusion as a source of energy

- Fusion offers an important potential source of energy on earth, since it could provide enormous amounts of energy from an almost inexhaustible energy source, and produce negligible pollution.
- Because of the tiny cross section, proton-proton reactions are useless. However, deuterium makes up 10^{-4} of all H on earth, and reactions such as $^2\text{H}(\text{d}, \text{n})^3\text{He}$ and $^2\text{H}(\text{d}, \text{p})^3\text{H}$ which release 3.3 and 4.0 MeV respectively may be usable. Alternatively, the deuterium–tritium reaction, which releases more than four times as much energy as a deuterium–deuterium reaction, has a larger reaction cross section, and uses easily produced tritium, may become practical.
- The plasmas in which controlled fusion occurs must have a temperature of the order of 20 keV (2×10^8 K) and hence must be confined in some non-mechanical

way (e.g. by magnetic fields) and heated very rapidly. It has turned out to be extremely difficult to do this in such a way as to extract more fusion energy than the energy initially expended in making fuel, confining it, and heating it. Heating and energy extraction have been particularly problematical. At present – after decades of experimentation – we are still apparently far from useful power extraction.

3 Late stages of stellar evolution

3.1 Stellar collapse and compact stars

- When a star has extracted all the available energy from the nuclear fuels found in the deep interior (where the temperature is high enough), it cannot continue to support itself by thermal pressure. In this case it inexorably shrinks, or even collapses.
- Two structures are available in nature in which the star can support itself *without* thermal pressure. These structures are available only at considerably higher mean densities than those found in main sequence stars. One, the *white dwarf* state, in which the star is supported by *electron* degeneracy pressure, can be reached through shrinkage. The other, the *neutron star* state, in which support pressure is provided by degenerate neutrons, seems only to be reachable following a catastrophic collapse. Such a collapse may be triggered by reaching such high density that capture of electrons at the top of the Fermi sea by protons to form neutrons becomes energetically advantageous.
- In both kinds of stars, pressure support is provided by the high energies forced on fermions at high density by the exclusion principle. We have already met this effect in action in atomic nuclei, where the population of successively higher energy levels is forced by this effect. We have seen that the total number of fermion states per unit volume available up to energy E is

$$\mathcal{N}(E)/V = \frac{1}{3\pi^2} \left(\frac{2mE}{\hbar^2} \right)^{3/2} .$$

Thus to increase the number of particles in a given volume, it is necessary to increase the maximum E_{\max} of states occupied (for non-relativistic particles) as $E_{\max} \sim n_{\text{part}}^{2/3}$ where $n_{\text{part}}^{2/3}$ is the number of particles of a specific type per unit volume. Since higher maximum energy (and also higher mean energy) means that the particles exert higher pressure – even if they are at $T = 0$ – a sufficiently high density may provide the pressure to support a star against gravity.

- For a white dwarf, minimizing the total energy (non-relativistic degenerate energy plus gravitational energy) for a given mass, by varying the radius (C & G do the algebra), gives a radius of roughly

$$R_{\min} \approx 7000(M_{\odot}/M)^{1/3} \text{ km},$$

where we assume $A/Z = 2$. Thus a white dwarf is about the size of the earth. The corresponding result for a neutron star replaces the 7000 km with 12.6 km; a $1M_{\odot}$ neutron star is about the size of a small asteroid.

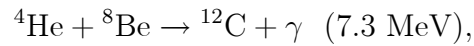
- Degenerate pressure support is however only conditionally available. If the density is high enough to make most of the particles relativistic, the pressure–density relationship changes and the pressure does not grow fast enough with increased density to support added weight. There is a limit – about 1.5 solar masses for white dwarfs, of order $3 M_{\odot}$ for neutron stars – above which the degenerate structure cannot support a star. If a star collapses to neutron star dimensions with too large a mass remaining, it becomes a black hole.

3.2 Nuclear reactions in late stellar evolution

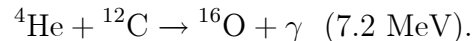
- Returning to stellar evolution: after H fuel in the centre of a star is exhausted, the core shrinks (and the envelope expands to create a giant star). Core shrinkage releases gravitational energy, increasing T_c to 10^8 K. ${}^4\text{He}$ begins to fuse



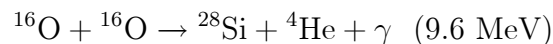
${}^8\text{Be}$ is unstable, but a tiny equilibrium population is established, making possible



so the unstable step in the synthesis chain is jumped over. At the same time alphas begin to react with the carbon:



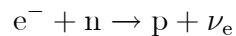
- When this fuel is exhausted, in a star of solar mass or so, electrons in the core become degenerate and the star becomes a white dwarf. In a more massive star contraction – and reactions – continue in ever more complex paths, for example



at about 10^9 K.

- As the temperature continues to rise, photons begin to have enough energy to break up nuclei, and *nuclear statistical equilibrium* is established, much like the equilibrium among various ionization states in a plasma at 10^4 K. Gammas destroy nuclei which reform by collisions, so that a whole range of intermediate nuclei form. Much iron is produced at this stage, but the destruction of heavy nuclei into lighter ones *costs* energy, so the pressure falls below the value needed to support the star. The density increases, the temperature rises, but at this point there is essentially no more nuclear energy to be gotten. The core of the star collapses to become a neutron star or black hole.

- The collapse releases an enormous amount of gravitational energy, about 10^2 MeV per particle. This is about 10% of the total rest mass of a proton or neutron, ten time more energy than has been liberated during the entire nuclear evolution up to this point. All the nuclear synthesis done in the core up to this point is undone. Somehow (it's still not really clear how) a good fraction of the energy is also transferred to the envelope of the star (which has not had time to collapse yet). The envelope is blown off at a speed of 10^4 km s⁻¹ or more.
- During the collapse, as the electrons are compressed to ever higher densities and the Fermi energy goes up and up, it becomes energetically advantageous for the electron capture reaction (*inverse beta decay*)



to occur, converting most of the protons into neutrons and setting the stage for creation of a neutron star (if the core mass is not too high), or a black hole. This reaction releases floods of neutrinos, enough so that when a supernova occurred in the Large Magellanic Cloud, a small galaxy at about 160,000 light-yr from us, about 20 of these neutrinos were detected by Kamiokande and the IMB detector in the USA.

- Another very interesting aspect of this kind of late nuclear evolution is that this is when the elements heavier than iron are created. Already during helium and carbon burning, such reactions as $^{13}\text{C}(\alpha, n)^{16}\text{O}$ produce neutrons of which many are absorbed by pre-existing nuclei of iron peak elements, gradually producing small numbers of heavier elements all the way up to $A > 200$ by slow neutron addition (the *s-process*). This neutron addition is slow in the sense that most beta-decays have time to occur between neutron additions, so as a nucleus evolves it stays close to the valley of beta stability.
- During a supernova explosion, violent heating of the envelope as it is ejected leads to many nuclear reactions, some of which release neutrons as well. In this setting, rapid neutron addition (the *r-process*) occurs, in which neutrons are added more rapidly than decays can occur, leading – temporarily – to some nuclei far from the line of stability. These decay back to the valley, of course, but some nuclei can only have been formed in this way.
- When the universe first formed, it contained only H, He and tiny amounts of Li, Be and B. All the heavier elements have formed either in evolving stars or in supernova explosions. The fact that elements up to iron form from reactions that occur in large regions of a star, while heavier elements form only by neutron addition, is the main reason that the relative abundances of chemical elements drop abruptly after the iron peak, from elements like Fe and Cr that are present in the solar system at a level of about 10^{-6} to 10^{-4} (by number) of H, while significantly heavier elements have relative number abundances nearer 10^{-10} compared to H.