Chapter 4. Lagrangian Dynamics

(Most of the material presented in this chapter is taken from Thornton and Marion, Chap. 7)

4.1 Important Notes on Notation

In this chapter, unless otherwise stated, the following notation conventions will be used:

1. Einstein's summation convention. Whenever an index appears twice (an only twice), then a summation over this index is implied. For example,

$$x_i x_i \equiv \sum_i x_i x_i = \sum_i x_i^2. \tag{4.1}$$

2. The index i is reserved for Cartesian coordinates. For example, x_i, for $i = 1,2,3$, represents either x, y, or z depending on the value of i. Similarly, p_i can represent p_x, p_y, or p_z. This does **not** mean that any other indices cannot be used for Cartesian coordinates, but that the index i will only be used for Cartesian coordinates.
3. When dealing with systems containing multiple particles, the index α will be used to identify quantities associated with a given particle when using Cartesian coordinates. For example, if we are in the presence of n particles, the position vector for particle α is given by $\mathbf{r}_\alpha$, and its kinetic energy T_α by

$$T_\alpha = \frac{1}{2} m_\alpha \dot{x}_{\alpha,i} \dot{x}_{\alpha,i}, \qquad \alpha = 1, 2, \ldots, n \text{ and } i = 1,2,3. \tag{4.2}$$

Take note that, according to convention 1 above, there is an implied summation on the Cartesian velocity components (the index i is used), but **not** on the masses since the index α appears more than twice. Correspondingly, the total kinetic energies is written as

$$T = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \dot{x}_{\alpha,i} \dot{x}_{\alpha,i} = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right). \tag{4.3}$$

4.2 Introduction

Although Newton's equation $\mathbf{F} = \dot{\mathbf{p}}$ correctly describes the motion of a particle (or a system of particles), it is often the case that a problem will be too complicated to solve using this formalism. For example, a particle may be restricted in its motion such that it follows the contours of a given surface, or that the forces that keep the particle on the surface (i.e., the **forces of constraints**), are not easily expressible in Cartesian coordinates. It may not even be possible at times to find expressions for some forces of constraints. Such occurrences would render it impossible to treat the problem with the Newtonian formalism since this requires the knowledge of all forces acting on the particles.

In this section we will study a different approach for solving complicated problems in a general manner. The formalism that will be introduced is based on the so-called **Hamilton's Principle**, from which the equations of motion will be derived. These equations are called **Lagrange's equations**. Although the method based on Hamilton's Principle does not constitute in itself a new physical theory, it is probably justified to say that it is more fundamental that Newton's equations. This is because Hamilton's Principle can be applied to a much wider range of physical phenomena than Newton's theory (e.g., quantum mechanics, quantum field theory, electromagnetism, relativity). However, as will be shown in the following sections, the Lagrange's equation derived from this new formalism are equivalent to Newton's equations when restricted to problems of mechanics.

4.3 Hamilton's Principle

Hamilton's Principle is concerned with the minimization of a quantity (i.e., the action) in a manner that is identical to extremum problems solved using the calculus of variations. Hamilton's Principle can be stated as follows:

> *The motion of a system from time t_1 to t_2 is such that the line integral (called the* action *or the* action integral*),*
>
> $$I = \int_{t_1}^{t_2} L\,dt, \tag{4.4}$$
>
> *where $L = T - U$ (with T,* and *U the kinetic and potential energies, respectively), has a stationary value for the actual path of the motion.*

Note that a "stationary value" for equation (4.4) implies an extremum for the action, not necessarily a minimum. But in almost all important applications in dynamics a minimum occurs.

Because of the dependency of the kinetic and potential energies on the coordinates x_i, the velocities $\dot{x}_i$, and possibly the time t, it is found that

$$L = L\left(x_i, \dot{x}_i, t\right). \tag{4.5}$$

Hamilton's Principle can now be expressed mathematically by

$$\boxed{\delta \int_{t_1}^{t_2} L\left(x_i, \dot{x}_i, t\right) dt = 0}. \tag{4.6}$$

Equation (4.6) can readily be solved by the technique described in the chapter on the calculus of variations. The solution is

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \qquad i = 1, 2, \ldots, n}. \tag{4.7}$$

Equations (4.7) are called the **Lagrange equations of motion**, and the quantity $L(x_i, \dot{x}_i, t)$ is the **Lagrangian**.

For example, if we apply Lagrange's equation to the problem of the one-dimensional harmonic oscillator (without damping), we have

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2, \tag{4.8}$$

and

$$\begin{aligned} \frac{\partial L}{\partial x} &= -kx \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{d}{dt}(m\dot{x}) = m\ddot{x}. \end{aligned} \tag{4.9}$$

After substitution of equations (4.9) into equation (4.7) we find

$$m\ddot{x} + kx = 0 \tag{4.10}$$

for the equation of motion. This result is identical than what was obtained using Newtonian mechanics. This is, however, a simple problem that can easily (and probably more quickly) be solved directly from the Newtonian formalism. But, the benefits of using the Lagrangian approach become obvious if we consider more complicated problems. For example, we try to determine the equations of motion of a particle of mass m constrained to move on the surface of a sphere under the influence of a conservative force $\mathbf{F} = F_\theta \mathbf{e}_\theta$, with F_θ a constant. In this case we have

$$\begin{aligned} T &= \frac{1}{2}m{v_\theta}^2 + \frac{1}{2}m{v_\phi}^2 \\ &= \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\sin^2(\theta)\dot{\phi}^2 \\ U &= -F_\theta R\theta, \end{aligned} \tag{4.11}$$

where we have defined the potential energy such that $U = 0$ when $\theta = \phi = 0$. The Lagrangian is given by

$$L = T - U = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \sin^2(\theta)\dot{\phi}^2 + F_\theta R\theta. \tag{4.12}$$

Upon inspection of the Lagrangian, we can see that there are two degrees of freedom for this problem, i.e., θ, and ϕ. We now need to calculate the different derivatives that compose the Lagrange equations

$$\begin{aligned}
&\frac{\partial L}{\partial \theta} = m R^2 \dot{\phi}^2 \sin(\theta)\cos(\theta) + F_\theta R \\
&\frac{\partial L}{\partial \phi} = 0 \\
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(m R^2 \dot{\theta}\right) = m R^2 \ddot{\theta} \\
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(m R^2 \dot{\phi} \sin^2(\theta)\right) = m R^2 \left(2\dot{\theta}\dot{\phi}\sin(\theta)\cos(\theta) + \ddot{\phi}\sin^2(\theta)\right),
\end{aligned} \tag{4.13}$$

applying equation (4.7) for θ, and ϕ we find the equations of motion to be

$$\begin{aligned}
F_\theta &= m R\left(\ddot{\theta} - \dot{\phi}^2 \sin(\theta)\cos(\theta)\right) \\
0 &= m R^2 \sin(\theta)\left(\ddot{\phi}\sin(\theta) + 2\dot{\theta}\dot{\phi}\cos(\theta)\right).
\end{aligned} \tag{4.14}$$

Incidentally, this problem was analyzed at the end of Chapter 1 on Newtonian Mechanics (problem 2-2, with $F_\phi = 0$), where Newton's equation was used to solve the problem. One can see how simpler the present treatment is. There was no need to calculate relatively complex equations like $\ddot{\mathbf{e}}_r$. Furthermore, it is important to realize that the spherical coordinates θ and ϕ are treated as Cartesian coordinates when using the Lagrangian formalism.

4.4 Degrees of Freedom and Generalized Coordinates

If a system is made up of n particles, we can specify the positions of all particles with $3n$ coordinates. On the other hand, if there are m equations of constraints (for example, if some particles were connected to form rigid bodies), then the $3n$ coordinates are not all independent. There will be only $3n - m$ independent coordinates, and the system is said to possess $3n - m$ *degrees of freedom*.

Furthermore, the coordinates and degrees of freedom do not have to be all given in Cartesian coordinates, or any other systems. In fact, we can choose to have different types of coordinate systems for different coordinates. Also, the degrees of freedom do not even need to share the same unit dimensions. For example, a problem with a mixture of Cartesian and spherical coordinates will have "lengths" and "angles" as units. Because of the latitude available in selecting the different degrees in freedom, the name of **generalized coordinates** is given to any set of quantities that completely specifies the state of the system. The generalized coordinates are usually written as q_j.

It is important to realize that there is not one unique way of setting up the generalized coordinates, there are indeed many different ways of doing this. Unfortunately, there are no clear rules for selecting the “best” set of generalized coordinates. The ultimate test is whether or not the choice made leads to a simple solution for the problem at hand.

In addition to the generalized coordinates q_j, a corresponding set of **generalized velocities** $\dot{q}_j$ is defined. In general, the relationships linking the Cartesian and generalized coordinates and velocities can be expressed as

$$\begin{aligned} x_{\alpha,i} &= x_{\alpha,i}\left(q_j,t\right) \\ \dot{x}_{\alpha,i} &= \dot{x}_{\alpha,i}\left(q_j,\dot{q}_j,t\right), \qquad \alpha = 1,\ \dots\ , n, \quad i = 1,2,3, \quad j = 1,\ \dots\ , 3n-m, \end{aligned} \tag{4.15}$$

or alternatively

$$\begin{aligned} q_j &= q_j\left(x_{\alpha,i},t\right) \\ \dot{q}_j &= \dot{q}_j\left(x_{\alpha,i},\dot{x}_{\alpha,i},t\right). \end{aligned} \tag{4.16}$$

We must also include the equations of constraints

$$f_k\left(x_{\alpha,i},t\right) = f_k\left(q_j,t\right) = 0. \tag{4.17}$$

It follows naturally that Hamilton’s Principle can now be expressed in term of the generalized coordinates and velocities as

$$\boxed{\delta\int_{t_1}^{t_2} L\left(q_j,\dot{q}_j,t\right)dt = 0}, \tag{4.18}$$

with Lagrange’s equations given by

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0, \qquad j = 1,2,\ \dots\ ,3n-m}. \tag{4.19}$$

Examples

1) *The simple pendulum*. Let’s solve the problem of the simple pendulum (of mass m and length ℓ) by first using the Cartesian coordinates to express the Lagrangian, and then transform into a system of cylindrical coordinates.

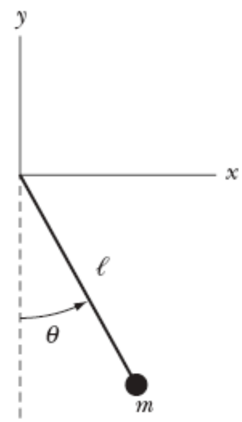


Figure 4-1 – A simple pendulum of mass *m* and length ℓ.

Solution. In Cartesian coordinates the kinetic and potential energies, and the Lagrangian are

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \\ U &= mgy \\ L &= T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy. \end{aligned} \tag{4.20}$$

We can now transform the coordinates with the following relations

$$\begin{aligned} x &= \ell\sin(\theta) \\ y &= -\ell\cos(\theta). \end{aligned} \tag{4.21}$$

Taking the time derivatives, we find

$$\begin{aligned} \dot{x} &= \ell\dot{\theta}\cos(\theta) \\ \dot{y} &= \ell\dot{\theta}\sin(\theta) \\ L &= \frac{1}{2}m\left(\ell^2\dot{\theta}^2\cos^2(\theta) + \ell^2\dot{\theta}^2\sin^2(\theta)\right) + mg\ell\cos(\theta) \\ &= \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos(\theta). \end{aligned} \tag{4.22}$$

We can now see that there is only one generalized coordinates for this problem, i.e., the angle θ. We can use equation (4.19) to find the equation of motion

$$
\begin{aligned}
&\frac{\partial L}{\partial \theta} = -mg\ell \sin(\theta) \\
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(m\ell^2\dot{\theta}\right) = m\ell^2\ddot{\theta},
\end{aligned} \tag{4.23}
$$

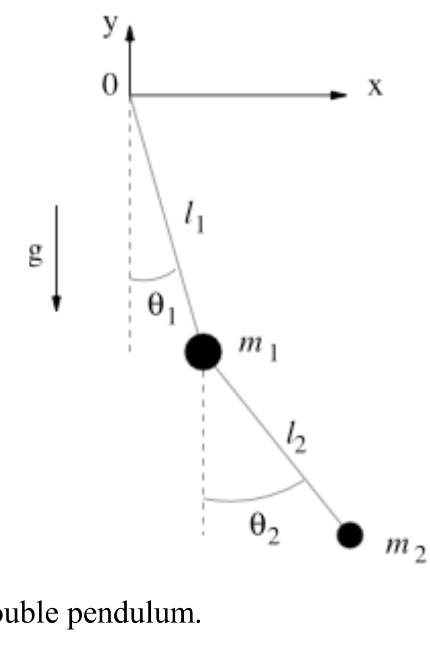
and finally

$$
\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0. \tag{4.24}
$$

2) *The double pendulum.* Consider the case of two particles of mass m_1 and m_2 each attached at the end of a mass less rod of length l_1 and l_2, respectively. Moreover, the second rod is also attached to the first particle (see Figure 4-2). Derive the equations of motion for the two particles.

Solution. It is desirable to use cylindrical coordinates for this problem. We have two degrees of freedom, and we will choose θ_1 and θ_2 as the independent variables. Starting with Cartesian coordinates, we write an expression for the kinetic and potential energies for the system

$$
\begin{aligned}
T &= \frac{1}{2}\left[m_1\left(\dot{x}_1^{\,2} + \dot{y}_1^{\,2}\right) + m_2\left(\dot{x}_2^{\,2} + \dot{y}_2^{\,2}\right)\right] \\
U &= m_1 g y_1 + m_2 g y_2.
\end{aligned} \tag{4.25}
$$



But Figure 4-2 – The double pendulum.

$$
\begin{aligned}
x_1 &= l_1 \sin(\theta_1) \\
y_1 &= -l_1 \cos(\theta_1) \\
x_2 &= l_1 \sin(\theta_1) + l_2 \sin(\theta_2) \\
y_2 &= -l_1 \cos(\theta_1) - l_2 \cos(\theta_2),
\end{aligned}
\tag{4.26}
$$

and

$$
\begin{aligned}
\dot{x}_1 &= l_1 \dot{\theta}_1 \cos(\theta_1) \\
\dot{y}_1 &= l_1 \dot{\theta}_1 \sin(\theta_1) \\
\dot{x}_2 &= l_1 \dot{\theta}_1 \cos(\theta_1) + l_2 \dot{\theta}_2 \cos(\theta_2) \\
\dot{y}_2 &= l_1 \dot{\theta}_1 \sin(\theta_1) + l_2 \dot{\theta}_2 \sin(\theta_2).
\end{aligned}
\tag{4.27}
$$

Inserting equations (4.26) and (4.27) in (4.25), we get

$$
\begin{aligned}
T &= \frac{1}{2}\Big[m_1 l_1^2 \dot{\theta}_1^2 \\
&\quad + m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \left\{ \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) \right\} \right) \Big] \\
&= \frac{1}{2}\left[m_1 l_1^2 \dot{\theta}_1^2 + m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right) \right] \\
U &= -(m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2),
\end{aligned}
\tag{4.28}
$$

and for the Lagrangian

$$
\begin{aligned}
L &= T - U \\
&= \frac{1}{2}\left[m_1 l_1^2 \dot{\theta}_1^2 + m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right) \right] \\
&\quad + (m_1 + m_2) g l_1 \cos(\theta_1) + m_2 g l_2 \cos(\theta_2).
\end{aligned}
\tag{4.29}
$$

Inspection of equation (4.29) tells us that there are two degrees of freedom for this problem, and we choose θ_1, and θ_2 as the corresponding generalized coordinates. We now use this Lagrangian with equation (4.19)

$$\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin(\theta_1) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} &= (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \right] \\
\frac{\partial L}{\partial \theta_2} &= m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin(\theta_2) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} &= m_2 \left(l_2^2 \ddot{\theta}_2 + l_1 l_2 \left[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \right] \right),
\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}
&(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \right] \\
&\qquad + (m_1 + m_2) g l_1 \sin(\theta_1) = 0 \\
&m_2 \left(l_2^2 \ddot{\theta}_2 + l_1 l_2 \left[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) \right] \right) + m_2 g l_2 \sin(\theta_2) = 0.
\end{aligned} \tag{4.31}$$

We can rewrite these equations as

$$\begin{aligned}
&\ddot{\theta}_1 + \frac{m_2}{m_1 + m_2} \frac{l_2}{l_1} \left(\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \right) + \frac{g}{l_1} \sin(\theta_1) = 0 \\
&\ddot{\theta}_2 + \frac{l_1}{l_2} \left(\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) \right) + \frac{g}{l_2} \sin(\theta_2) = 0.
\end{aligned} \tag{4.32}$$

3) *The pendulum on a rotating rim*. A simple pendulum of length b and mass m moves on a mass-less rim of radius a rotating with constant angular velocity ω (see Figure 4-3). Get the equation of motion for the mass.

Solution. If we choose the center of the rim as the origin of the coordinate system, we calculate

$$\begin{aligned}
x &= a \cos(\omega t) + b \sin(\theta) \\
y &= a \sin(\omega t) - b \cos(\theta),
\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}
\dot{x} &= -a\omega \sin(\omega t) + b\dot{\theta} \cos(\theta) \\
\dot{y} &= a\omega \cos(\omega t) + b\dot{\theta} \sin(\theta).
\end{aligned} \tag{4.34}$$

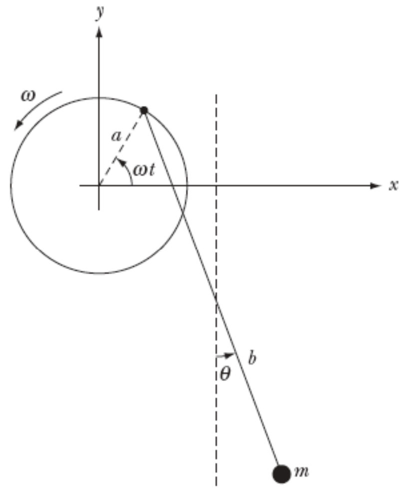


Figure 4-3 – A simple pendulum attached to a rotating rim.

The kinetic and potential energies, and the Lagrangian are

$$\begin{aligned}
T &= \frac{1}{2}m\left(a^2\omega^2 + b^2\dot{\theta}^2 + 2ab\omega\dot{\theta}\left[\sin(\theta)\cos(\omega t) - \sin(\omega t)\cos(\theta)\right]\right) \\
&= \frac{1}{2}m\left(a^2\omega^2 + b^2\dot{\theta}^2 + 2ab\omega\dot{\theta}\sin(\theta - \omega t)\right) \\
U &= mg\left(a\sin(\omega t) - b\cos(\theta)\right) \\
L &= T - U \\
&= \frac{1}{2}m\left(a^2\omega^2 + b^2\dot{\theta}^2 + 2ab\omega\dot{\theta}\sin(\theta - \omega t)\right) - mg\left(a\sin(\omega t) - b\cos(\theta)\right).
\end{aligned} \tag{4.35}$$

We now calculate the derivatives for the Lagrange equation using θ as the sole generalized coordinate

$$\begin{aligned}
\frac{\partial L}{\partial \theta} &= mab\omega\dot{\theta}\cos(\theta - \omega t) - mgb\sin(\theta) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= mb^2\ddot{\theta} + mab\omega\left(\dot{\theta} - \omega\right)\cos(\theta - \omega t).
\end{aligned} \tag{4.36}$$

Finally, the equation of motion is

$$\ddot{\theta} - \frac{a}{b}\omega^2\cos(\theta - \omega t) + \frac{g}{b}\sin(\theta - \omega t) = 0. \tag{4.37}$$

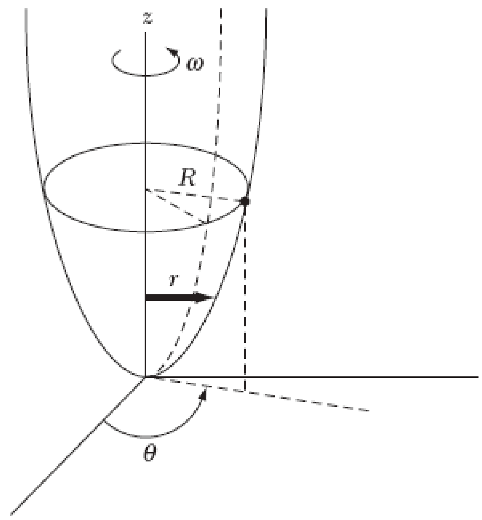


Figure 4-4 – A slides along a smooth wire that rotates about the z-axis.

4) *The sliding bead*. A bead slides along a smooth wire that has the shape of a parabola $z = cr^2$ (see Figure 4-4). At equilibrium, the bead rotates in a circle of radius R when the wire is rotating about its vertical symmetry axis with angular velocity ω. Find the value of c.

Solution. We choose the cylindrical coordinates $r, \theta,$ and z as generalized coordinates. The kinetic and potential energies are

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) \\ U &= mgz. \end{aligned} \tag{4.38}$$

We have in this case some equations of constraints that we must take into account, namely

$$\begin{aligned} z &= cr^2 \\ \dot{z} &= 2c\dot{r}r, \end{aligned} \tag{4.39}$$

and

$$\begin{aligned} \theta &= \omega t \\ \dot{\theta} &= \omega. \end{aligned} \tag{4.40}$$

Inserting equations (4.39) and (4.40) in equation (4.38), we can calculate the Lagrangian for the problem

$$
\begin{aligned}
L &= T - U \\
&= \frac{1}{2} m\left(\dot{r}^2 + 4c^2 r^2 \dot{r}^2 + r^2\omega^2\right) - mgcr^2.
\end{aligned}
\tag{4.41}
$$

It is important to note that the inclusion of the equations of constraints in the Lagrangian has reduced the number of degrees of freedom to only one, i.e., r. We now calculate the equation of motion using Lagrange's equation

$$
\begin{aligned}
\frac{\partial L}{\partial r} &= m\left(4c^2 r\dot{r}^2 + r\omega^2 - 2gcr\right) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= m\left(\ddot{r} + 4c^2 r^2 \ddot{r} + 8c^2 r\dot{r}^2\right),
\end{aligned}
\tag{4.42}
$$

and

$$
\ddot{r}\left(1 + 4c^2 r^2\right) + \dot{r}^2\left(4c^2 r\right) + r\left(2gc - \omega^2\right) = 0. \tag{4.43}
$$

When the bead is in equilibrium, we have $r = R$ and $\dot{r} = \ddot{r} = 0$, and equation (4.43) reduces to

$$
R\left(2gc - \omega^2\right) = 0, \tag{4.44}
$$

or

$$
c = \frac{\omega^2}{2g}. \tag{4.45}
$$

4.5 Lagrange's Equations with Undetermined Multipliers

A system that is subjected to holonomic constraints (i.e., constraints that can be expressed in the form $f\left(x_{\alpha,i},t\right) = f\left(q_j,t\right) = 0$) will always allow the selection of a proper set of generalized coordinates for which the equations of motion will be free of the constraints themselves. Alternatively, constraints that are functions of the velocities, and which can be written in a differential form and integrated to yield relations amongst the coordinates are also holonomic. For example, an equation of the form

$$
A_i \frac{dx_i}{dt} + B = 0 \tag{4.46}
$$

cannot, in general, be integrated to give an equation of the form $f\left(x_i,t\right) = 0$. Such equations of constraints are non-holonomic. We will not consider this type of constraints any further. However, if the constants A_i, and B are such that equation (4.46) can be expressed as

$$\frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t}+\frac{\partial f}{\partial t}=0, \tag{4.47}$$

or more simply

$$\frac{df}{dt}=0, \tag{4.48}$$

then it can be integrated to give

$$f\left(x_i,t\right)-cste=0. \tag{4.49}$$

Using generalized coordinates, and by slightly changing the form of equation (4.48), we conclude that, as was stated above, constraints that can be written in the form of a differential

$$df=\frac{\partial f}{\partial q_j}dq_j+\frac{\partial f}{\partial t}dt=0 \tag{4.50}$$

are similar to the constraints considered at the beginning of this section, that is

$$f\left(q_j,t\right)-cste=0. \tag{4.51}$$

Problems involving constraints such as the holonomic kind discussed here can be handled in exactly the same manner as was done in the chapter on the calculus of variations. This is done by introducing the so-called Lagrange undetermined multipliers. When this is done, we find that the following form for the Lagrange equations

$$\boxed{\frac{\partial L}{\partial q_j}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}+\lambda_k\left(t\right)\frac{\partial f_k}{\partial q_j}=0} \tag{4.52}$$

where the index $j=1,2,\ldots,3n-m,$ and $k=1,2,\ldots,m$.

Although the Lagrangian formalism does not require the insertion of the forces of constraints involved in a given problem, these forces are closely related to the Lagrange undetermined multipliers. The corresponding generalized forces of constraints can be expressed as

$$Q_j=\lambda_k\left(t\right)\frac{\partial f_k}{\partial q_j}. \tag{4.53}$$

Examples

1) *The rolling disk on an inclined plane*. We now solve the problem of a disk of mass m and of radius R rolling down an inclined plane (see Figure 4-5).

Solution. Referring to Figure 4-1, and separating the kinetic energies in a translational rotational part, we can write

$$\begin{aligned} T &= \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 \\ &= \frac{1}{2}m\dot{y}^2 + \frac{1}{4}mR^2\dot{\theta}^2, \end{aligned} \tag{4.54}$$

where $I = mR^2/2$ is the moment of inertia of the disk about its axis of rotation. The potential energy and the Lagrangian are given by

$$\begin{aligned} U &= mg(l-y)\sin(\alpha) \\ L &= T - U \\ &= \frac{1}{2}m\dot{y}^2 + \frac{1}{4}mR^2\dot{\theta}^2 - mg(l-y)\sin(\alpha), \end{aligned} \tag{4.55}$$

where l, and α are the length and the angle of the inclined plane, respectively. The equation of constraint given by

$$f = y - R\theta = 0. \tag{4.56}$$

This problem presents itself with two generalized coordinates (y and θ) and one equation of constraints, which leaves us with one degree of freedom. We now apply the Lagrange equations as defined with equation (4.52)

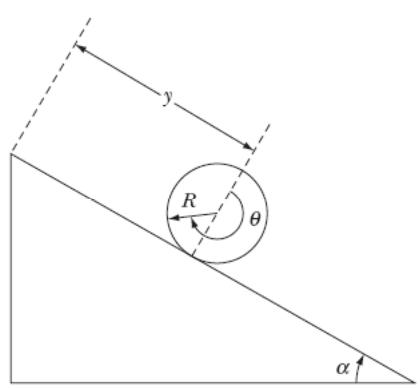


Figure 4-5 - A disk rolling on an incline plane without slipping.

$$\begin{aligned}\frac{\partial L}{\partial y}-\frac{d}{dt}\frac{\partial L}{\partial \dot{y}}+\lambda\frac{\partial f}{\partial y}&=mg\sin(\alpha)-m\ddot{y}+\lambda=0\\ \frac{\partial L}{\partial \theta}-\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}+\lambda\frac{\partial f}{\partial \theta}&=-\frac{1}{2}mR^2\ddot{\theta}-\lambda R=0.\end{aligned}\tag{4.57}$$

From the last equation we have

$$\lambda=-\frac{1}{2}mR\ddot{\theta},\tag{4.58}$$

which using the equation of constraint (4.56) may be written as

$$\lambda=-\frac{1}{2}m\ddot{y}.\tag{4.59}$$

Inserting this last expression in the first of equation (4.57) we find

$$\ddot{y}=\frac{2}{3}g\sin(\alpha),\tag{4.60}$$

and

$$\lambda=-\frac{1}{3}mg\sin(\alpha).\tag{4.61}$$

In a similar fashion we also find that

$$\ddot{\theta}=\frac{2g\sin(\alpha)}{3R}.\tag{4.62}$$

Equations (4.60) and (4.62) can easily be integrated, and the forces of constraints that keep the disk from sliding can be evaluated from equation (4.53)

$$\begin{aligned}Q_y&=\lambda\frac{\partial f}{\partial y}=\lambda=-\frac{1}{3}mg\sin(\alpha)\\ Q_\theta&=\lambda\frac{\partial f}{\partial \theta}=\lambda R=-\frac{1}{3}mRg\sin(\alpha).\end{aligned}\tag{4.63}$$

Take note that Q_y and Q_θ are a force and a torque, respectively. This justifies the appellation of generalized forces.

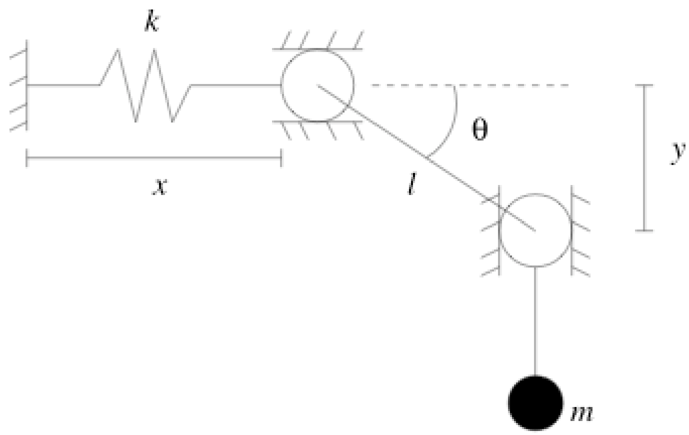


Figure 4-6 – An arrangement of a spring, mass, and mass less pulleys.

2) Consider the system of Figure 4-6. A string joining two mass less pulleys has a length of l and makes an angle θ with the horizontal. This angle will vary has a function of the vertical position of the mass. The two pulleys are restricted to a translational motion by frictionless guiding walls. The restoring force of the spring is $-kx$ and the arrangement is such that when $\theta = 0$, $x = 0$, and when $\theta = \pi/2$, $x = l$. Find the equations of motion for the mass.

Solution. There are two equations of constraints for this problem

$$\begin{aligned} f &= x - l\big(1 - \cos(\theta)\big) = 0 \\ g &= y - l\sin(\theta) = 0, \end{aligned} \tag{4.64}$$

and we identify three generalized coordinates $x, y,$ and θ. We now calculate the energies and the Lagrangian

$$\begin{aligned} T &= \frac{1}{2} m\dot{y}^2 \\ U &= \frac{1}{2} kx^2 - mgy \\ L &= T - U = \frac{1}{2} m\dot{y}^2 - \frac{1}{2} kx^2 + mgy. \end{aligned} \tag{4.65}$$

Using equations (4.64) and (4.65), we can write the Lagrange equations while using the appropriate undetermined Lagrange multipliers

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \lambda_f \frac{\partial f}{\partial q_j} + \lambda_g \frac{\partial g}{\partial q_j}, \tag{4.66}$$

where q_j can represent $x, y,$ or θ. Calculating the necessary derivatives

$$
\begin{aligned}
&\frac{\partial L}{\partial x} = -kx \\
&\frac{\partial L}{\partial y} = mg \\
&\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \\
&\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = m\ddot{y},
\end{aligned}
\tag{4.67}
$$

and

$$
\begin{aligned}
&-kx + \lambda_1 = 0 \\
&mg - m\ddot{y} + \lambda_2 = 0 \\
&-\lambda_1 l \sin(\theta) - \lambda_2 l \cos(\theta) = 0.
\end{aligned}
\tag{4.68}
$$

We, therefore, have

$$
\begin{aligned}
\lambda_1 &= kx = kl\left(1 - \cos(\theta)\right) = kl\left(1 - \sqrt{1 - \left(\frac{y}{l}\right)^2}\right) \\
\lambda_2 &= -\lambda_1 \tan(\theta) \\
&= -kl\left(1 - \sqrt{1 - \left(\frac{y}{l}\right)^2}\right) \cdot \frac{y}{l\sqrt{1 - \left(\frac{y}{l}\right)^2}} \\
&= -ky\left(\left[1 - \left(\frac{y}{l}\right)^2\right]^{-\frac{1}{2}} - 1\right),
\end{aligned}
\tag{4.69}
$$

and with the insertion of these equations in the second of equations (4.68) we finally get

$$
\ddot{y} - g + \frac{k}{m} y\left(\left[1 - \left(\frac{y}{l}\right)^2\right]^{-\frac{1}{2}} - 1\right) = 0.
\tag{4.70}
$$

4.6 Equivalence of Lagrange's and Newton's Equations

In this section we will prove the equivalence of the Lagrangian and the Newtonian formalisms of mechanics. We consider the simple case where the generalized coordinates are the Cartesian coordinates, and we concentrate on the dynamics of a single particle not subjected to forces of constraints. The Lagrange equation for this problem is

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \qquad i = 1,2,3. \tag{4.71}$$

Since $L = T - U$ and $T = T(\dot{x}_i)$, and $U = U(x_i)$ for a conservative system (e.g., for a particle falling vertically in a gravitational field we have $T = m\dot{y}^2/2$, and $U = mgy$), Lagrange's equation becomes

$$-\frac{\partial U}{\partial x_i} = \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i}, \tag{4.72}$$

since

$$\frac{\partial T}{\partial x_i} = \frac{\partial U}{\partial \dot{x}_i} = 0. \tag{4.73}$$

For a conservative system we also have

$$F_i = -\frac{\partial U}{\partial x_i}, \tag{4.74}$$

and

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} = \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\left(\frac{1}{2}m\dot{x}_k\dot{x}_k\right) = \frac{d}{dt}(m\dot{x}_i) = \dot{p}_i, \tag{4.75}$$

where p_i is component i of the momentum.

From equations (4.74) and (4.75) we finally obtain

$$F_i = \dot{p}_i, \tag{4.76}$$

which are, of course, the Newtonian equations of motion.

4.7 Conservation Theorems

Before deriving the usual conservation theorem using the Lagrangian formalism, we must first consider how we can express the kinetic energy as a function of the generalized coordinates and velocities.

4.7.1 The Kinetic Energy

In a Cartesian coordinates system the kinetic energy of a system of particles is expressed as

$$T = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \dot{x}_{\alpha,i}\dot{x}_{\alpha,i}, \tag{4.77}$$

where a summation over i is implied. In order to derive the corresponding relation using generalized coordinates and velocities, we go back to the first of equations (4.15), which relates the two systems of coordinates

$$x_{\alpha,i} = x_{\alpha,i}\left(q_j, t\right), \qquad j = 1, 2, \ldots, 3n - m. \tag{4.78}$$

Taking the time derivative of this equation we have

$$\dot{x}_{\alpha,i} = \frac{\partial x_{\alpha,i}}{\partial q_j}\dot{q}_j + \frac{\partial x_{\alpha,i}}{\partial t}, \tag{4.79}$$

and squaring it (and summing over i)

$$\dot{x}_{\alpha,i}\dot{x}_{\alpha,i} = \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}\dot{q}_j\dot{q}_k + 2\frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial t}\dot{q}_j + \frac{\partial x_{\alpha,i}}{\partial t}\frac{\partial x_{\alpha,i}}{\partial t}. \tag{4.80}$$

An important case occurs when a system is *scleronomic*, i.e., there is no explicit dependency on time in the coordinate transformation, we then have

$$\frac{\partial x_{\alpha,i}}{\partial t} = 0,$$

and the kinetic energy can be written in the form

$$\boxed{T = a_{jk}\dot{q}_j\dot{q}_k} \tag{4.81}$$

with

$$a_{jk} = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}, \tag{4.82}$$

where a summation on i is still implied. Just as was the case for Cartesian coordinates, we see that the kinetic energy is a quadratic function of the (generalized) velocities. If we next differentiate equation (4.81) with respect to $\dot{q}_l$, and then multiply it by $\dot{q}_l$ (and summing), we get

$$\boxed{\dot{q}_l \frac{\partial T}{\partial \dot{q}_l} = 2a_{jk}\dot{q}_j\dot{q}_k = 2T} \tag{4.83}$$

since a_{jk} is not a function of the generalized velocities, and it is symmetric in the exchange of the j and k indices.

4.7.2 Conservation of Energy

Consider a general Lagrangian, which will be a function of the generalized coordinates and velocities and may also depend explicitly on time (this dependence may arise from time variation of external potentials, or from time-dependent constraints). Then the total time derivative of L is

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t}. \tag{4.84}$$

But from Lagrange's equations,

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}, \tag{4.85}$$

and equation (4.84) can be written as

$$\begin{aligned} \frac{dL}{dt} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t} \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) + \frac{\partial L}{\partial t}. \end{aligned} \tag{4.86}$$

It therefore follows that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L\right) + \frac{\partial L}{\partial t} = \frac{dH}{dt} + \frac{\partial L}{\partial t} = 0 \tag{4.87}$$

or

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}, \tag{4.88}$$

where we have introduced a new function

$$H = \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L. \tag{4.89}$$

In cases where the Lagrangian is not explicitly dependent on time we find that

$$H = \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = cste. \tag{4.90}$$

If we are in presence of a scleronomic system, where there is also no explicit time dependence in the coordinate transformation (i.e., $x_{\alpha,i} = x_{\alpha,i}\left(q_j\right)$), then $U = U\left(q_j\right)$ and $\partial U / \partial \dot{q}_j = 0$ and

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial \left(T - U\right)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}. \tag{4.91}$$

Equation (4.90) can be written as

$$\begin{aligned} H &= \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - L \\ &= 2T - L \\ &= T + U = E = cste, \end{aligned} \tag{4.92}$$

where we have used the result obtained in equation (4.83) for the second line.

The function H is called the **Hamiltonian** of the system and it is equaled to the total energy only if the following conditions are met:

1. The equations of the transformation connecting the Cartesian and generalized coordinates must be independent of time (the kinetic energy is then a quadratic function of the generalized velocities).
2. The potential energy must be velocity independent.

It is important to realize that these conditions may not always be realized. For example, if the total energy is conserved in a system, but that the transformation from Cartesian to generalized coordinates involve time (i.e., a moving generalized coordinate system), then equations (4.81) and (4.83) don't apply and the Hamiltonian expressed in the moving system does not equal the energy. We are in a presence of a case where the total energy is conserved, but the Hamiltonian is not.

4.7.3 Noether's Theorem

We can derive conservation theorems by taking advantage of the so-called Noether's theorem, which connects a given symmetry to the invariance of a corresponding physical quantity.

Consider a set of variations δq_j on the generalized coordinates that define a system, which may or may not be independent. We write the variation of the Lagrangian as

$$\delta L = \frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j, \tag{4.93}$$

but from the Lagrange equations we have

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right), \tag{4.94}$$

and

$$\begin{aligned} \delta L &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\delta q_j\right). \end{aligned} \tag{4.95}$$

Nother's theorem states that any set of variations δq_j (or symmetry) that leaves the Lagrangian of a system invariant (i.e., $\delta L = 0$) implies the conservation of the following quantity (from equation (4.95))

$$\frac{\partial L}{\partial \dot{q}_j}\delta q_j = cste. \tag{4.96}$$

4.7.4 Conservation of Linear Momentum

Consider the translation in space of an entire system. That is to say, every generalized coordinates is translated by an infinitesimal amount such that

$$q_j \to q_j + \delta q_j. \tag{4.97}$$

Because space is homogeneous in an inertial frame, the Lagrangian function of a closed system must be invariant when subjected to such a translation of the system in space. Therefore,

$$\delta L = 0. \tag{4.98}$$

Equation (4.96) then applies and

$$\frac{\partial L}{\partial \dot{q}_j}\delta q_j = cste \tag{4.99}$$

or

$$\frac{\partial L}{\partial \dot{q}_j} = cste, \tag{4.100}$$

since the displacements δq_j are arbitrary and independent. Let's further define a new function

$$p_j = \frac{\partial L}{\partial \dot{q}_j}. \tag{4.101}$$

Because of the fact that p_j reduces to an "ordinary" component of the linear momentum when dealing with Cartesian coordinates through

$$\begin{aligned} p_{\alpha,i} &= \frac{\partial L}{\partial \dot{x}_{\alpha,i}} = \frac{\partial (T - U)}{\partial \dot{x}_{\alpha,i}} \\ &= \frac{\partial T}{\partial \dot{x}_{\alpha,i}} = \frac{\partial \left(\frac{1}{2} m_\alpha \dot{x}_{\alpha,i} \dot{x}_{\alpha,i} \right)}{\partial \dot{x}_{\alpha,i}} = m_\alpha \dot{x}_{\alpha,i}, \end{aligned} \tag{4.102}$$

they are called **generalized momenta**. Inserting equation (4.101) into equation (4.100) we find

$$p_j = cste. \tag{4.103}$$

The generalized momentum component p_j is conserved. When dealing with Cartesian coordinates the total linear momentum p_i in a given direction is also conserved. That is, from equation (4.102) we have

$$p_i = \sum_{\alpha=1}^{n} p_{\alpha,i} = \sum_{\alpha=1}^{n} m_\alpha \dot{x}_{\alpha,i} = cste, \tag{4.104}$$

since $p_{\alpha,i} = cste$ for all α.

Furthermore, whenever equation (4.100) applies (or equivalently $\partial L / \partial q_j = 0$), it is said that the generalized coordinate q_j is *cyclic*. We then find the corresponding generalized momentum component p_j to be a constant of motion.

4.7.5 Conservation of Angular Momentum

Since space is isotropic, the properties of a closed system are unaffected by its orientation. In particular, the Lagrangian will be unaffected if the system is rotated through a small angle. Therefore,

$$\delta L = \frac{d}{dt}\left(p_j \delta q_j\right) = 0. \tag{4.105}$$

Let's now consider the case where the Lagrangian is expressed as a function of Cartesian coordinates such that we can make the following substitutions

$$\begin{aligned} q_j &\to x_{\alpha,i} \\ p_j &\to p_{\alpha,i} = m_\alpha \dot{x}_{\alpha,i}. \end{aligned} \tag{4.106}$$

Referring to Figure 4-7, we can expressed the variation $\delta \mathbf{r}_\alpha$ in the position vector $\mathbf{r}_\alpha$ caused by an infinitesimal rotation $\delta \boldsymbol{\theta}$ as

$$\delta \mathbf{r}_\alpha = \delta \boldsymbol{\theta} \times \mathbf{r}_\alpha \tag{4.107}$$

Inserting equations (4.106) and (4.107) in equation (4.105) we have

$$\begin{aligned} \frac{d}{dt}\left(p_{\alpha,i} \delta x_{\alpha,i}\right) &= \frac{d}{dt}\left(\mathbf{p}_\alpha \cdot \delta \mathbf{r}_\alpha\right) \\ &= \frac{d}{dt}\left[\mathbf{p}_\alpha \cdot \left(\delta \boldsymbol{\theta} \times \mathbf{r}_\alpha\right)\right] \\ &= \frac{d}{dt}\left[\delta \boldsymbol{\theta} \cdot \left(\mathbf{r}_\alpha \times \mathbf{p}_\alpha\right)\right] \\ &= \delta \boldsymbol{\theta} \cdot \frac{d}{dt}\left[\left(\mathbf{r}_\alpha \times \mathbf{p}_\alpha\right)\right], \end{aligned} \tag{4.108}$$

where we used the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$. We further transform equation (4.108) to

$$\delta \boldsymbol{\theta} \cdot \frac{d}{dt}\left(\mathbf{r}_\alpha \times \mathbf{p}_\alpha\right) = \delta \boldsymbol{\theta} \cdot \frac{d}{dt}\mathbf{L}_\alpha = 0, \tag{4.109}$$

with $\mathbf{L}_\alpha = \mathbf{r}_\alpha \times \mathbf{p}_\alpha$ is the angular momentum vector associated with the particle identified with the index α. But since $\delta \boldsymbol{\theta}$ is arbitrary we must have

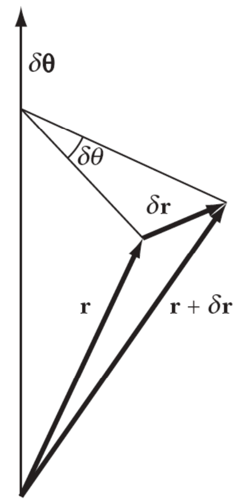


Figure 4-7 – A system rotated by an infinitesimal amount $\delta\theta$.

$$\mathbf{L}_\alpha = cste, \tag{4.110}$$

for all α. Finally, summing over α we find that the total angular momentum $\mathbf{L}$ is conserved, that is

$$\mathbf{L} = \sum_{\alpha=1}^{n} \left(\mathbf{r}_\alpha \times \mathbf{p}_\alpha \right) = cste. \tag{4.111}$$

It is important to note that although we only used Noether's theorem to prove the conservation of the linear and angular momenta, it can also be used to express the conservation of energy. But to do so requires a relativistic treatment where time is treated on equal footing with the other coordinates (i.e., $x_{\alpha,i}$ or q_j).

4.8 D'Alembert's Principle and Lagrange's Equations

(This section is optional and will not be subject to examination. It will not be found in Thornton and Marion. The treatment presented below closely follows that of Goldstein, Poole and Safko, pp. 16-21.)

It is important to realize that Lagrange's equations were not originally derived using Hamilton's Principle. Lagrange himself placed the subject on a sound mathematical foundation by using the concept of *virtual work* along with *D'Alembert's Principle*.

4.8.1 Virtual Work and D'Alembert's Principle

A virtual displacement is the result of any infinitesimal change of the coordinates $\delta\mathbf{r}_\alpha$ that define a particular system, and which is *consistent with the different forces and constraints imposed on the system at a given instant* t. The term virtual is used to distinguish these types of displacement with *actual* displacement occurring in a time interval dt, during which the forces could be changing.

Now, suppose that a system is in equilibrium. In that case the total force $\mathbf{F}_\alpha$ on each particle that compose the system must vanish, i.e., $\mathbf{F}_\alpha = 0$. If we define the virtual work done on a particle as $\mathbf{F}_\alpha \bullet \delta \mathbf{r}_\alpha$ (note that we are using Cartesian coordinates) then we have

$$\sum_\alpha \mathbf{F}_\alpha \bullet \delta \mathbf{r}_\alpha = 0. \tag{4.112}$$

Let's now decompose the force $\mathbf{F}_\alpha$ as the sum of the applied force $\mathbf{F}_\alpha^{(a)}$ and the force of constraint $\mathbf{f}_\alpha$ such that

$$\mathbf{F}_\alpha = \mathbf{F}_\alpha^{(a)} + \mathbf{f}_\alpha, \tag{4.113}$$

then equation (4.112) becomes

$$\sum_\alpha \mathbf{F}_\alpha^{(a)} \bullet \delta \mathbf{r}_\alpha + \sum_\alpha \mathbf{f}_\alpha \bullet \delta \mathbf{r}_\alpha = 0. \tag{4.114}$$

In what follows, we will restrict ourselves to systems where the net virtual work of the forces of constraints is zero, that is

$$\sum_\alpha \mathbf{f}_\alpha \bullet \delta \mathbf{r}_\alpha = 0, \tag{4.115}$$

and

$$\sum_\alpha \mathbf{F}_\alpha^{(a)} \bullet \delta \mathbf{r}_\alpha = 0. \tag{4.116}$$

This condition will hold for many types of constraints. For example, if a particle is forced to move on a surface, the force of constraint if perpendicular to the surface while the virtual displacement is tangential. It is, however, not the case for sliding friction forces since they are directed against the direction of motion; we must exclude them from our analysis. But for systems where the force of constraints are consistent with equation (4.115), then equation (4.116) is valid and is referred to as the **principle of virtual work**.

Now, let's consider the equation of motion $\mathbf{F}_\alpha = \dot{\mathbf{p}}_\alpha$, which can be written as

$$\mathbf{F}_\alpha - \dot{\mathbf{p}}_\alpha = 0. \tag{4.117}$$

Inserting this last equation in equation (4.112) we get

$$\sum_\alpha \left(\mathbf{F}_\alpha - \dot{\mathbf{p}}_\alpha \right) \bullet \delta \mathbf{r}_\alpha = 0, \tag{4.118}$$

and upon using equations (4.113) and (4.115) we find

$$\sum_{\alpha}\left(\mathbf{F}_{\alpha}^{(a)}-\dot{\mathbf{p}}_{\alpha}\right)\bullet\delta\mathbf{r}_{\alpha}=0. \tag{4.119}$$

Equation (4.119) is often called *D'Alembert's Principle*.

4.8.2 Lagrange's Equations

We now go back to our usual coordinate transformation that relates the Cartesian and generalized coordinates

$$x_{\alpha,i}=x_{\alpha,i}\left(q_j,t\right), \tag{4.120}$$

from which we get

$$\frac{dx_{\alpha,i}}{dt}=\frac{\partial x_{\alpha,i}}{\partial q_j}\dot{q}_j+\frac{\partial x_{\alpha,i}}{\partial t}. \tag{4.121}$$

Similarly, the components $\delta x_{\alpha,i}$ of the virtual displacements vectors can be written as

$$\delta x_{\alpha,i}=\frac{\partial x_{\alpha,i}}{\partial q_j}\delta q_j. \tag{4.122}$$

Note that no time variation δt is involved in equation (4.122) since, by definition, a virtual displacement is defined as happening at a given instant t, and not within a time interval δt. Inserting equation (4.122) in the first term of equation (4.119), we have

$$\begin{aligned}\sum_{\alpha}F_{\alpha,i}^{(a)}\delta r_{\alpha,i}&=\sum_{\alpha}F_{\alpha,i}^{(a)}\frac{\partial x_{\alpha,i}}{\partial q_j}\delta q_j\\&=Q_j\delta q_j,\end{aligned} \tag{4.123}$$

where summations on i, and j are implied, and the quantity

$$Q_j=\sum_{\alpha}F_{\alpha,i}^{(a)}\frac{\partial x_{\alpha,i}}{\partial q_j} \tag{4.124}$$

are the components of the **generalized forces**.

Concentrating now on the second term of equation (4.119) we write

$$\sum_{\alpha}\dot{p}_{\alpha,i}\delta x_{\alpha,i}=\sum_{\alpha}m_{\alpha}\ddot{x}_{\alpha,i}\frac{\partial x_{\alpha,i}}{\partial q_j}\delta q_j. \tag{4.125}$$

This last equation can be rewritten as

$$\sum_{\alpha} m_{\alpha}\ddot{x}_{\alpha,i}\frac{\partial x_{\alpha,i}}{\partial q_j}\delta q_j = \sum_{\alpha}\left[\frac{d}{dt}\left(m_{\alpha}\dot{x}_{\alpha,i}\frac{\partial x_{\alpha,i}}{\partial q_j}\right) - m_{\alpha}\dot{x}_{\alpha,i}\frac{d}{dt}\left(\frac{\partial x_{\alpha,i}}{\partial q_j}\right)\right]\delta q_j. \tag{4.126}$$

We can modify the last term since

$$\frac{d}{dt}\left(\frac{\partial x_{\alpha,i}}{\partial q_j}\right) = \frac{\partial \dot{x}_{\alpha,i}}{\partial q_j}. \tag{4.127}$$

Furthermore, we can verify from equation (4.121) that

$$\frac{\partial \dot{x}_{\alpha,i}}{\partial \dot{q}_k} = \frac{\partial x_{\alpha,i}}{\partial q_k}. \tag{4.128}$$

Substituting equations (4.127) and (4.128) into (4.126) leads to

$$\begin{aligned}\sum_{\alpha} m_{\alpha}\ddot{x}_{\alpha,i}\frac{\partial x_{\alpha,i}}{\partial q_j}\delta q_j &= \sum_{\alpha}\left[\frac{d}{dt}\left(m_{\alpha}\dot{x}_{\alpha,i}\frac{\partial \dot{x}_{\alpha,i}}{\partial \dot{q}_j}\right) - m_{\alpha}\dot{x}_{\alpha,i}\frac{\partial \dot{x}_{\alpha,i}}{\partial q_j}\right]\delta q_j \\ &= \sum_{\alpha}\left[\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m_{\alpha}\dot{x}_{\alpha,i}\dot{x}_{\alpha,i}\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{1}{2}m_{\alpha}\dot{x}_{\alpha,i}\dot{x}_{\alpha,i}\right)\right]\delta q_j.\end{aligned} \tag{4.129}$$

Combining this last result with equations (4.124) and (4.125), we can express (4.119) as

$$\sum_{\alpha}\left(F_{\alpha,i}^{(a)} - \dot{p}_{\alpha,i}\right)\delta x_{\alpha,i} = \left\{Q_j - \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\right\}\delta q_j = 0, \tag{4.130}$$

where we have introduced T the kinetic energy of the system, such that

$$T = \frac{1}{2}\sum_{\alpha} m_{\alpha}\dot{x}_{\alpha,i}\dot{x}_{\alpha,i}. \tag{4.131}$$

Since the set of virtual displacements δq_j are independent, the only way for equation (4.130) to hold is that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j. \tag{4.132}$$

If we now limit ourselves to conservative systems, we must have

$$F_{\alpha,i}^{(a)} = -\frac{\partial U}{\partial x_{\alpha,i}}, \tag{4.133}$$

and similarly,

$$\begin{aligned} Q_j = F_{\alpha,i}^{(a)} \frac{\partial x_{\alpha,i}}{\partial q_j} &= -\frac{\partial U}{\partial x_{\alpha,i}} \frac{\partial x_{\alpha,i}}{\partial q_j} \\ &= -\frac{\partial U}{\partial q_j}, \end{aligned} \tag{4.134}$$

with $U = U\left(x_{\alpha,i},t\right) = U\left(q_j,t\right)$ the potential energy. We can, therefore, rewrite equation (4.132) as

$$\frac{d}{dt}\left(\frac{\partial\left(T-U\right)}{\partial \dot{q}_j}\right) - \frac{\partial\left(T-U\right)}{\partial q_j} = 0, \tag{4.135}$$

since $\partial U/\partial \dot{q}_j = 0$.

If we now define the Lagrangian function for the system as

$$L = T - U, \tag{4.136}$$

we finally recover Lagrange's equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}. \tag{4.137}$$

4.8.3 Dissipative Forces and Rayleigh's Dissipative Function

So far, we have only dealt with system where there is no dissipation of energy. Lagrange's equations can, however, be made to accommodate some of these situations. To see how this can be done, we will work our way backward from Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0. \tag{4.138}$$

If we allow for the generalized forces on the system Q_j to be expressible in the following manner

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right), \tag{4.139}$$

then equation (4.138) can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j. \tag{4.140}$$

We now allow for some frictional forces, which cannot be derived from a potential such as expressed in equation (4.139), but for example, are expressed as follows

$$f_j = -k_j \dot{q}_j, \tag{4.141}$$

where **no** summation on the repeated index is implied. Expanding our definition of generalized forces to include the friction forces

$$Q_j \to Q_j + f_j, \tag{4.142}$$

Equation (4.140) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) + f_j, \tag{4.143}$$

or, alternatively

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = f_j. \tag{4.144}$$

Dissipative forces of the type shown in equation (4.141) can be derived in term of a function R, known as **Rayleigh's dissipation function**, and defined as

$$R = \frac{1}{2}\sum_j k_j \dot{q}_j^{\,2}. \tag{4.145}$$

From this definition it is clear that

$$f_j = -\frac{\partial R}{\partial \dot{q}_j}, \tag{4.146}$$

and the Lagrange equations with dissipation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}+\frac{\partial R}{\partial \dot{q}_,}=0, \tag{4.147}$$

so that two scalar functions, L and R, must be specified to obtain the equations of motion.

4.8.4 Velocity-dependent Potentials

Although we exclusively studied potentials that have no dependency on the velocities, the Lagrangian formalism is well suited to handle some systems where such potentials arise. This is the case, for example, when the generalized forces can be expressed with equation (4.139). That is,

$$Q_j=-\frac{\partial U}{\partial q_j}+\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right). \tag{4.148}$$

This equation applies to a very important type of force field, namely, the electromagnetic forces on moving charges.

Consider an electric charge, q, of mass m moving at velocity $\mathbf{v}$ in a region subjected to an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$, which may both depend on time and position. As is known from electromagnetism theory, the charge will experience the so-called Lorentz force

$$\mathbf{F}=q\left[\mathbf{E}+\left(\mathbf{v}\times\mathbf{B}\right)\right]. \tag{4.149}$$

Both the electric and the magnetic fields are derivable from a scalar potential ϕ and a vector potential $\mathbf{A}$ by

$$\mathbf{E}=-\nabla\phi-\frac{\partial \mathbf{A}}{\partial t}, \tag{4.150}$$

and

$$\mathbf{B}=\nabla\times\mathbf{A}. \tag{4.151}$$

The Lorentz force on the charge can be obtained if the velocity-dependent potential energy U is expressed

$$U=q\phi-q\mathbf{A}\bullet\mathbf{v}, \tag{4.152}$$

so that the Lagrangian is

$$L = T - U$$
$$= \frac{1}{2}mv^2 - q\phi + q\mathbf{A} \bullet \mathbf{v}. \tag{4.153}$$

4.9 The Lagrangian Formulation for Continuous Systems

4.9.1 The Transition from a Discrete to a Continuous System

Let's consider the case of an infinite elastic rod that can undergo small longitudinal vibrations. A system composed of discrete particles that approximate the continuous rod is an infinite chain of equal mass points spaced a distance a apart and connected by uniform mass less springs having force constants k (see Figure 4-8).

Denoting the displacement of the jth particle from its equilibrium position by η_j, the kinetic and potential energies can be written as

$$T = \frac{1}{2}\sum_j m\dot{\eta}_j^2$$
$$U = \frac{1}{2}\sum_j k\left(\eta_{j+1} - \eta_j\right)^2. \tag{4.154}$$

The Lagrangian is then given by

$$L = T - U$$
$$= \frac{1}{2}\sum_j \left[m\dot{\eta}_j^2 - k\left(\eta_{j+1} - \eta_j\right)^2\right], \tag{4.155}$$

which can also be written as

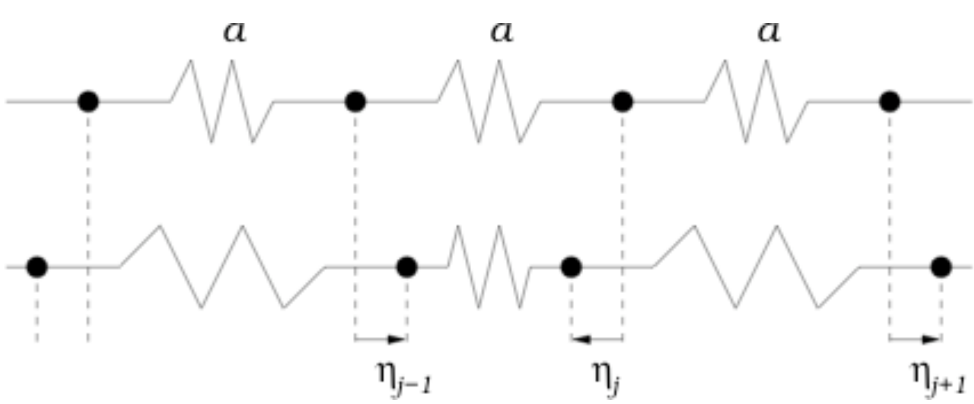


Figure 4-8 – A discrete system of equal mass springs connected by springs, as an approximation to a continuous elastic rod.

$$L = \frac{1}{2}\sum_j a\left[\frac{m}{a}\dot{\eta}_j^{\,2} - ka\left(\frac{\eta_{j+1}-\eta_j}{a}\right)^2\right] = \sum_j aL_j, \qquad (4.156)$$

where a is the equilibrium separation between the particles and L_j is the quantity contained in the square brackets. The particular form of the Lagrangian given in equation (4.156) was chosen so that we can easily go to the limit of a continuous rod as a approaches zero. In going from the discrete to the continuous case, the index j becomes the continuous position coordinate x, and we therefore have

$$\lim_{a\to 0}\frac{\eta_{j+1}-\eta_j}{a} = \frac{\eta(x+a)-\eta(x)}{a} = \frac{d\eta}{dx}, \qquad (4.157)$$

where a takes on the role of dx. Furthermore, we have

$$\begin{aligned} &\lim_{a\to 0}\frac{m}{a} = \mu \\ &\lim_{a\to 0} ka = Y, \end{aligned} \qquad (4.158)$$

where μ is the mass per unit length and Y is Young's modulus (note that in the continuous case Hooke's Law becomes $F = -Y\, d\eta/dx$). We can also impose the same limit to the Lagrangian of equation (4.156) while taking equations (4.157) and (4.158) into account. We then obtain

$$L = \frac{1}{2}\int\left[\mu\dot{\eta}^2 - Y\left(\frac{d\eta}{dx}\right)^2\right]dx. \qquad (4.159)$$

This simple example illustrates the main features of passing from a discrete to a continuous system. The most important thing to grasp is the role played by the position coordinates x. It is **not** a generalized coordinates, but it now takes on the role of being a parameter in the same right as the time is. The generalized coordinate is the variable $\eta = \eta(x,t)$. If the continuous system were three-dimensional, then we would have $\eta = \eta(x,y,z,t)$, where $x, y, z,$ and t would be completely independent of each other. We can generalize the Lagrangian for the three-dimensional system as

$$L = \iiint \mathcal{L}\, dx\, dy\, dz, \qquad (4.160)$$

where $\mathcal{L}$ is the **Lagrangian density**. In the example of the one-dimensional continuous elastic rod considered above we have

$$\mathcal{L} = \frac{1}{2}\left[\mu\dot{\eta}^2 - Y\left(\frac{d\eta}{dx}\right)^2\right]. \tag{4.161}$$

4.9.2 The Lagrange Equations of Motion for Continuous Systems

We will now derive the Lagrange equations of motion for the case of a one-dimensional continuous system. The extension to a three-dimensional system is straightforward. The Lagrangian density in this case is given by

$$\mathcal{L} = \mathcal{L}\left(\eta, \frac{d\eta}{dx}, \frac{d\eta}{dt}, x, t\right). \tag{4.162}$$

We now apply Hamilton's Principle to the action integral in a way similar to what was done for discrete systems

$$\delta I = \delta\int_{t_1}^{t_2}\int_{x_1}^{x_2} \mathcal{L}\,dx\,dt = 0. \tag{4.163}$$

We then propagate the variation using the shorthand δ notation introduced in section 3.3

$$\begin{aligned}\delta I &= \int_{t_1}^{t_2}\int_{x_1}^{x_2} \delta\mathcal{L}\,dx\,dt \\ &= \int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\frac{\partial\mathcal{L}}{\partial\eta}\delta\eta + \frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta}{dx}\right)}\delta\left(\frac{d\eta}{dx}\right) + \frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta}{dt}\right)}\delta\left(\frac{d\eta}{dt}\right)\right]dx\,dt,\end{aligned} \tag{4.164}$$

and since $\delta(d\eta/dx) = d(\delta\eta)/dx$ and $\delta(d\eta/dt) = d(\delta\eta)/dt$, we have

$$\delta I = \int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\frac{\partial\mathcal{L}}{\partial\eta}\delta\eta + \frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta}{dx}\right)}\frac{d(\delta\eta)}{dx} + \frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta}{dt}\right)}\frac{d(\delta\eta)}{dt}\right]dx\,dt. \tag{4.165}$$

Integrating the last two terms on the right hand side by parts we finally get

$$\delta I = \int_{t_1}^{t_2}\int_{x_1}^{x_2}\left\{\frac{\partial\mathcal{L}}{\partial\eta} - \frac{d}{dx}\left[\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta}{dx}\right)}\right] - \frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta}{dt}\right)}\right]\right\}\delta\eta\,dx\,dt = 0. \tag{4.166}$$

Since the virtual variation $\delta\eta$ is arbitrary, we have for the Lagrange equations of motion of a continuous system

$$\boxed{\frac{\partial \mathcal{L}}{\partial \eta} - \frac{d}{dx}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{d\eta}{dx}\right)}\right] - \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{d\eta}{dt}\right)}\right] = 0} \tag{4.167}$$

Applying equation (4.167) to our previous Lagrangian density for the elastic rod (i.e., equation (4.161)), we get

$$\mu \frac{d^2\eta}{dt^2} - Y\frac{d^2\eta}{dx^2} = 0. \tag{4.168}$$

This is the so-called one-dimensional **wave equation**, which has for a general solution

$$\eta(x,t) = f(x+vt) + g(x-vt), \tag{4.169}$$

where f and g are two arbitrary functions of $x+vt$ and $x-vt$, and $v = \sqrt{Y/\mu}$.