Chapter 8. Motion in a Noninertial Reference Frame

(Most of the material presented in this chapter is taken from Thornton and Marion, Chap. 10.)

We have so far dealt only with problems situated in inertial reference frame, or if not, problems that could be solved with enough accuracy by ignoring the noninertial nature of the coordinate systems. There are, however, many problems for which it is necessary, or beneficial, to treat the motion of the system at hand in a noninertial reference frame. In this chapter, we will develop the mathematical apparatus that will allow us to deal with such problems, and prepare the way for the study of the motion of rigid bodies that we will tackle in the next chapter.

8.1 Rotating Coordinate Systems

Let's consider two coordinate systems: one that is inertial and for which the axes are *fixed*, and another whose axes are *rotating* with respect to the inertial system. We represent the coordinates of the fixed system by x_i' and the coordinates of the rotating system by x_i. If we choose some point in space P (see Figure 8-1) we have

$$\mathbf{r}' = \mathbf{R} + \mathbf{r}, \tag{8.1}$$

where $\mathbf{R}$ locates the origin of the rotating system in the fixed system. We assume that P is at rest in the inertial so that $\mathbf{r}'$ is constant.

If during an infinitesimal amount of time dt the rotating system undergoes an infinitesimal rotation $d\boldsymbol{\theta}$ about some axis, then the vector $\mathbf{r}$ will vary not only as measured by an observer co-moving with the rotating system, but also when measured in the inertial frame.

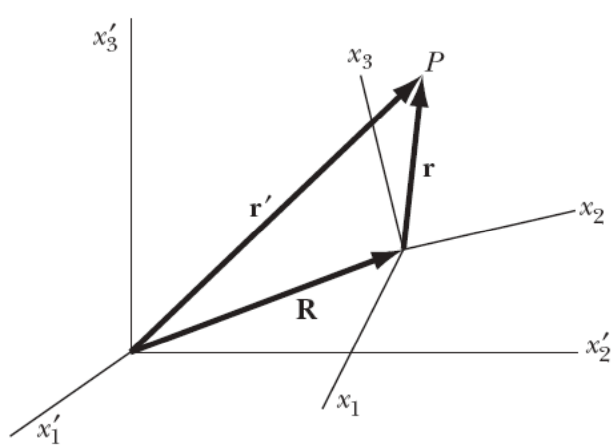

Figure 8-1 – The x_i''s are coordinates in the fixed system, and x_i are coordinates in the rotating system. The vector $\mathbf{R}$ locates the origin of the rotating system in the fixed system.

This problem is the same as was treated in Chapter 4 when considering the conservation of angular momentum using Noether's Theorem (see Figure 4.7 and equation (4.107) on page 79), and we can write

$$\left(d\mathbf{r}\right)_{\text{fixed}} = d\boldsymbol{\theta} \times \mathbf{r}, \tag{8.2}$$

where the designation "fixed" is included to indicate that $d\mathbf{r}$ is measured in the fixed or inertial coordinate system. We can obtain the time rate of change of $\mathbf{r}$ in the inertial system by dividing both sides of equation (8.2) by dt

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \boldsymbol{\omega} \times \mathbf{r}, \tag{8.3}$$

with

$$\boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}. \tag{8.4}$$

If we allow the point P to have some velocity $\left(d\mathbf{r}/dt\right)_{\text{rotating}}$ with respect to the rotating system, equation (8.3) must be correspondingly modified to account for this motion. Then, we have

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{r}. \tag{8.5}$$

Example

We have a vector $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ in a rotating system, which share a common origin with an inertial system. Find $\dot{\mathbf{r}}'$ in the fixed system by direct differentiation if the angular velocity of the rotating system is $\boldsymbol{\omega}$ in the fixed system.

Solution.
We have

$$\dot{\mathbf{r}}' = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \frac{d}{dt}\left(x_i\mathbf{e}_i\right) = \dot{x}\mathbf{e}_i + x_i\dot{\mathbf{e}}_i, \tag{8.6}$$

where a summation over a repeated index is implied. The first term on the right hand side of equation (8.6) is simply the velocity as measured in the rotating system (i.e., we have the components $\dot{x}_i$ along the corresponding axes $\mathbf{e}_i$, which form the basis vectors of the rotating system). We therefore rewrite equation (8.6) as

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + x_i\dot{\mathbf{e}}_i. \tag{8.7}$$

We need to evaluate $\dot{\mathbf{e}}_i$ for $i = 1,\ 2,\ \text{and}\ 3$. To do so, consider three infinitesimal rotations $d\theta_1,\ d\theta_2,\ \text{and}\ d\theta_3$ along $\mathbf{e}_1,\ \mathbf{e}_2,\ \text{and}\ \mathbf{e}_3$, respectively. If we first calculate the effect of the first rotation about $\mathbf{e}_1$ on the other two basis vectors (using Figure 8-2) we have

$$\begin{aligned} d\mathbf{e}_2 &= \left[\cos\left(d\theta_1\right)\mathbf{e}_2 + \sin\left(d\theta_1\right)\mathbf{e}_3\right] - \mathbf{e}_2 \\ &\simeq \left[\mathbf{e}_2 + d\theta_1\mathbf{e}_3\right] - \mathbf{e}_2 \\ &\simeq d\theta_1\mathbf{e}_3, \end{aligned} \tag{8.8}$$

since $d\theta_1$ is infinitesimal. Similarly, if we calculate the effect of each infinitesimal rotations on every basis vectors we find, after a division by dt, that

$$\begin{aligned} \frac{d\mathbf{e}_1}{dt} &= \omega_3\mathbf{e}_2 - \omega_2\mathbf{e}_3 \\ \frac{d\mathbf{e}_2}{dt} &= -\omega_3\mathbf{e}_1 + \omega_1\mathbf{e}_3 \\ \frac{d\mathbf{e}_3}{dt} &= \omega_2\mathbf{e}_1 - \omega_1\mathbf{e}_2, \end{aligned} \tag{8.9}$$

with $\omega_i = d\theta_i/dt$. Alternatively, we can combine equations (8.9) into one vector equation as

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i, \tag{8.10}$$

with $\boldsymbol{\omega} = \omega_i\mathbf{e}_i$. Inserting equation (8.10) into equation (8.7), we get

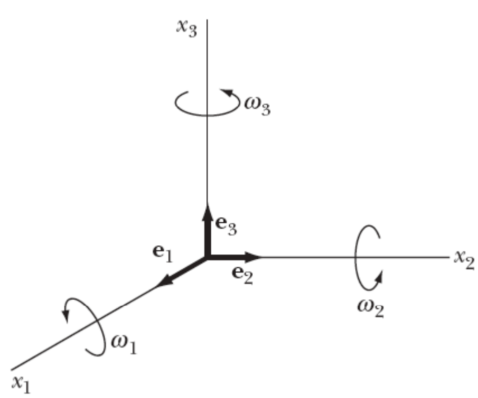

Figure 8-2 – With this definition for the set of axes, and with $\omega_i = d\theta_i/dt$, for $i = 1,\ 2,\ \text{and}\ 3$, we can determine the effect of the rotations on the different basis vectors.

$$
\begin{aligned}
\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} &= \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \left(\boldsymbol{\omega}\times x_i\mathbf{e}_i\right) \\
&= \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \left(\boldsymbol{\omega}\times\mathbf{r}\right),
\end{aligned}
\tag{8.11}
$$

which is the same result as equation (8.5).

8.1.1 **Generalization to arbitrary vectors**

Although we used the position vector $\mathbf{r}$ for the derivation of equation (8.11) (or (8.5)), this expression applies equally well to an arbitrary vector $\mathbf{Q}$, that is

$$
\boxed{\left(\frac{d\mathbf{Q}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega}\times\mathbf{Q}}
\tag{8.12}
$$

For example, we can verify that the angular acceleration $\dot{\boldsymbol{\omega}}$ is the same in both systems of reference

$$
\begin{aligned}
\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{fixed}} &= \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega}\times\boldsymbol{\omega} \\
&= \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rotating}}.
\end{aligned}
\tag{8.13}
$$

We can also use equation (8.12) to find the velocity of point P (in Figure 8-1) as measured in the fixed system

$$
\begin{aligned}
\left(\frac{d\mathbf{r}'}{dt}\right)_{\text{fixed}} &= \left(\frac{d\mathbf{R}}{dt}\right)_{\text{fixed}} + \left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} \\
&= \left(\frac{d\mathbf{R}}{dt}\right)_{\text{fixed}} + \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega}\times\mathbf{r}.
\end{aligned}
\tag{8.14}
$$

If we define the following quantities

$$
\begin{aligned}
\mathbf{v}_f &\equiv \dot{\mathbf{r}}_f \equiv \left(\frac{d\mathbf{r}'}{dt}\right)_{\text{fixed}} \\
\mathbf{V} &\equiv \dot{\mathbf{R}}_f \equiv \left(\frac{d\mathbf{R}}{dt}\right)_{\text{fixed}} \\
\mathbf{v}_r &\equiv \dot{\mathbf{r}}_r \equiv \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}},
\end{aligned}
\tag{8.15}
$$

we can rewrite equation (8.14) as

$$\boxed{\mathbf{v}_f = \mathbf{V} + \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}} \tag{8.16}$$

where

$$\begin{aligned}
\mathbf{v}_f &= \text{ the velocity relative to the fixed axes} \\
\mathbf{V} &= \text{ the linear velocity of the moving origin} \\
\mathbf{v}_r &= \text{ the velocity to the rotating axes} \\
\boldsymbol{\omega} &= \text{ the angular velocity of the rotating axes} \\
\boldsymbol{\omega} \times \mathbf{r} &= \text{ the velocity due to the rotation of the moving axes.}
\end{aligned} \tag{8.17}$$

8.2 The Centrifugal and Coriolis Forces

We know that Newton's Second Law (i.e., $\mathbf{F} = m\mathbf{a}$) is valid only in an inertial frame of reference. In other words, the simple form $\mathbf{F} = m\mathbf{a}$ for the equation of motion applies when the acceleration is that which is measured in the fixed referenced system, i.e., $\mathbf{a} \equiv \mathbf{a}_f$. Then, we can write

$$\mathbf{F} = m\mathbf{a}_f = m\left(\frac{d\mathbf{v}_f}{dt}\right)_{\text{fixed}}, \tag{8.18}$$

where the differentiation is carried out in the fixed system. Differentiating equation (8.16) we get

$$\left(\frac{d\mathbf{v}_f}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{V}}{dt}\right)_{\text{fixed}} + \left(\frac{d\mathbf{v}_r}{dt}\right)_{\text{fixed}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}}. \tag{8.19}$$

Using equation (8.12) we can transform this equation as follows

$$\begin{aligned}
\mathbf{a}_f &= \ddot{\mathbf{R}}_f + \left[\left(\frac{d\mathbf{v}_r}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{v}_r\right] + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \left[\left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{r}\right] \\
&= \ddot{\mathbf{R}}_f + \mathbf{a}_r + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}),
\end{aligned} \tag{8.20}$$

where $\ddot{\mathbf{R}}_f = (d\mathbf{V}/dt)_{\text{fixed}}$. Correspondingly, the force on the particle as measured in the inertial frame becomes

$$\mathbf{F} = m\ddot{\mathbf{R}}_f + m\mathbf{a}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\boldsymbol{\omega} \times \mathbf{v}_r. \tag{8.21}$$

Alternatively, the effective force on the particle as seen by an observer co-moving with the rotating system is

$$\begin{aligned}\mathbf{F}_{\mathrm{eff}} &\equiv m\mathbf{a}_r \\ &= \mathbf{F} - m\ddot{\mathbf{R}}_f - m\dot{\boldsymbol{\omega}}\times\mathbf{r} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - 2m\boldsymbol{\omega}\times\mathbf{v}_r.\end{aligned} \tag{8.22}$$

The first term is the total force acting on the particle as measured in the inertial frame. The second ($-m\ddot{\mathbf{R}}_f$) and third ($-m\dot{\boldsymbol{\omega}}\times\mathbf{r}$) are due to the translational and angular accelerations, respectively, of the moving noninertial system. The fourth term ($-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$) is the so-called **centrifugal force** (directed away from the centre of rotation), and finally, the last term ($-2m\boldsymbol{\omega}\times\mathbf{v}_r$) is the **Coriolis force**. It is important to note that the Coriolis force arises because of the motion of the particle in the rotating system, i.e., it disappears if $v_r = 0$.

Equation (8.22) is a mathematical representation of what is meant by the statement that Newton's Second Law does not apply in a noninertial reference frame. It is not that the physics dealt with Newtonian mechanics cannot be analyzed in a noninertial frame, but that the form of the equations of motion is different. More precisely, if we set $\ddot{\mathbf{R}}_f$ and $\dot{\boldsymbol{\omega}}$ in equation (8.22) to zero to simplify things, we have in the rotating frame a more complicated equation of motion

$$\mathbf{F}_{\mathrm{eff}} = m\mathbf{a}_r + \text{ (noninertial terms)}, \tag{8.23}$$

where the "noninertial terms" are the centrifugal and Coriolis forces, than in an inertial frame where the equation of motion is simply

$$\mathbf{F} = m\mathbf{a}_f. \tag{8.24}$$

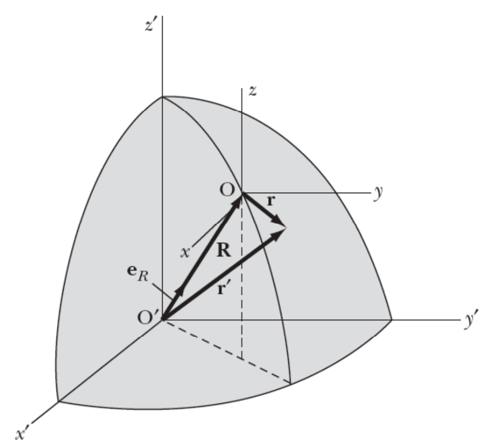

Figure 8-3 – The inertial reference system $x'y'z'$ has its origin O′ at the centre of the Earth, and the moving frame xyz has its centre near the Earth's surface. The vector $\mathbf{R}$ gives the Earth's radius.

8.3 Motion relative to the Earth

We can apply the results obtained in the previous section to motion near the surface of the Earth. If we set the origin of the inertial (fixed) system $x'y'z'$ to be at the center of the Earth, and the moving (rotating) noninertial frame xyz on the surface of the Earth, we can describe the motion of a moving object near its surface using equation (8.22). We denote by $\mathbf{F} = \mathbf{S} + m\mathbf{g}_0$ the total force acting on the object (of mass m) where $\mathbf{S}$ represent any external forces (except gravity) and $\mathbf{g}_0$ is the gravitational acceleration

$$\mathbf{g}_0 = -\frac{GM_\oplus}{R^2}\mathbf{e}_R. \tag{8.25}$$

In equation (8.25) $G = 6.67\times10^{-11}\,\mathrm{N\cdot m^2/kg^2}$ is the universal gravitational constant, $M_\oplus = 5.98\times10^{24}$ kg is the mass of the Earth, and $R = 6.38\times10^6$ m its radius (see Figure 8-3). We assume that the Earth's radius and gravitational field are independent of latitude. The effective force $\mathbf{F}_{\text{eff}}$ as measured in the moving frame near the surface of the Earth becomes

$$\mathbf{F}_{\text{eff}} = \mathbf{S} + m\mathbf{g}_0 - m\ddot{\mathbf{R}}_f - m\dot{\boldsymbol{\omega}}\times\mathbf{r} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - 2m\boldsymbol{\omega}\times\mathbf{v}_r. \tag{8.26}$$

The Earth's angular velocity vector is given by $\boldsymbol{\omega} = 7.3\times10^{-5}\mathbf{e}_{z'}$ rad/s (i.e., it is directed along the z'-axis), and we assume that it is a constant. The fourth term on the right hand side of equation (8.26) therefore equals zero. Also, from equation (8.12) we have

$$\begin{aligned}
\ddot{\mathbf{R}}_f &= \left(\frac{d\dot{\mathbf{R}}_f}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega}\times\dot{\mathbf{R}}_f \\
&= \boldsymbol{\omega}\times\left\{\boldsymbol{\omega}\times\left[\left(\frac{d\mathbf{R}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega}\times\mathbf{R}\right]\right\} \\
&= \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R}),
\end{aligned} \tag{8.27}$$

since $\mathbf{R}$ is a constant. Inserting equation (8.27) in equation (8.26) we get

$$\mathbf{F}_{\text{eff}} = \mathbf{S} + m\mathbf{g}_0 - m\boldsymbol{\omega}\times\left[\boldsymbol{\omega}\times(\mathbf{r}+\mathbf{R})\right] - 2m\boldsymbol{\omega}\times\mathbf{v}_r. \tag{8.28}$$

The second and third terms on the right hand side of this equation can be combined into a single term for the effective gravitational acceleration $\mathbf{g}$ that is felt near the surface of the Earth (i.e., on the surface of the Earth we cannot discerned between gravity $\mathbf{g}_0$ and the centrifugal acceleration $\boldsymbol{\omega}\times\left[\boldsymbol{\omega}\times(\mathbf{r}+\mathbf{R})\right]$, we can only feel the resulting acceleration $\mathbf{g}$)

$$\mathbf{g} = \mathbf{g}_0 - \boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{R}) \right]. \tag{8.29}$$

It is to be noted that because of the presence of the centrifugal acceleration $-\boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{R}) \right]$ in this equation for the effective gravity, $\mathbf{g}$ and $\mathbf{g}_0$ will in general not point exactly in the same direction. This effect is rather small, but measurable as $\omega R^2 / g_0 = 0.0035$. It should also be clear from the equation (8.29) that the magnitude of the effect is a function of latitude.

The equation for the effective force is then rewritten as

$$\boxed{\mathbf{F}_{\text{eff}} = \mathbf{S} + m\mathbf{g} - 2m\boldsymbol{\omega} \times \mathbf{v}_r} \tag{8.30}$$

As was pointed out earlier, the last term on the right hand side of equation (8.30) is responsible for the Coriolis effect. This effect is the source for some well-known motions of the air masses. To see how this happens, let's consider the xyz coordinate system to be located at some latitude λ where the angular velocity vector $\boldsymbol{\omega}$ (which represents the Earth's rotation) has a component $\omega_z \mathbf{e}_z$ along the vertical at the specified latitude. If a particle is projected such that its velocity vector $\mathbf{v}_r$ is located in the xy plane, then the Coriolis force will have a component directed to the right of the particle's motion (see Figure 8-4). The size of this effect will be a function of the latitude, as the amplitude of ω_z also exhibits such a dependency. So, consider a region where, for some reason, the atmospheric pressure is lower than it is in its surrounding (see Figure 8-5). As the air flows into this low-pressure spot from regions of higher pressure all around, the Coriolis effect will deflect the air motion to the right (in the Northern Hemisphere), resulting into counterclockwise, or cyclonic, motions in the atmosphere.

As the following example will show, the Coriolis effect generally only becomes important for the motion of bodies near the surface of the Earth when large enough distance scales are considered.

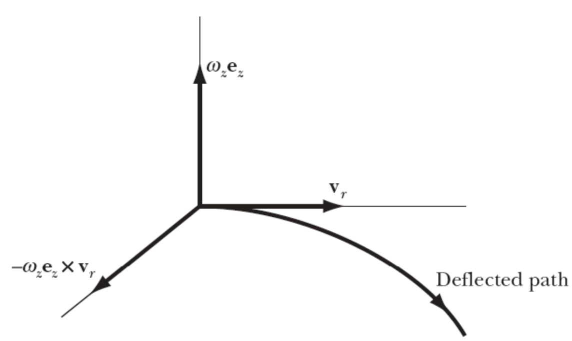

Figure 8-4 – In the Northern hemisphere, a particle projected in a horizontal plane will be directed to the right of its motion. The opposite will happen in the Southern Hemisphere.

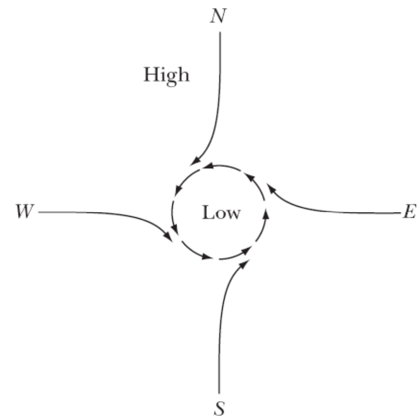

Figure 8-5 – The Coriolis effect deflects the air in the Northern Hemisphere to the right producing cyclonic motion.

Examples

1. Free-falling object. Find the horizontal deflection caused by the Coriolis effect acting on a free-falling particle in the Earth's effective gravitational field from a height $h(\ll R)$ above its surface.

Solution.
From equation (8.30), with $\mathbf{S}=0$ and $\mathbf{F}_{\text{eff}} = m\mathbf{a}_r$, we have

$$\mathbf{a}_r = \mathbf{g} - 2\boldsymbol{\omega}\times\mathbf{v}_r. \tag{8.31}$$

We choose the z-axis attached (virtually) to the surface of the rotating Earth as directed outward along $-\mathbf{g}$. We also choose the $\mathbf{e}_x$ and $\mathbf{e}_y$ bases vectors such that they are in the southerly and easterly direction, respectively. The latitude is once again denoted by λ (see Figure 8-6). With these definitions we can decompose the Earth's angular velocity vector as

$$\begin{aligned}\omega_x &= -\omega\cos(\lambda)\\ \omega_y &= 0\\ \omega_z &= \omega\sin(\lambda).\end{aligned} \tag{8.32}$$

Even though the Coriolis effect produces velocity components along $\mathbf{e}_x$ and $\mathbf{e}_y$, we will neglect these since they will be significantly smaller than the velocity along $-\mathbf{e}_z$. Then,

$$\begin{aligned}\dot{x} &\simeq \dot{y} \simeq 0\\ \dot{z} &\simeq -gt,\end{aligned} \tag{8.33}$$

where we assume that the particle is free-falling from rest.

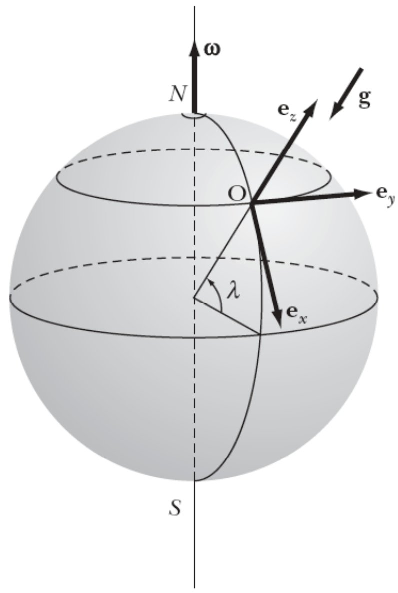

Figure 8-6 – The coordinated system "attached" to the Earth's surface, for finding the horizontal deflection of a free-falling particle. The $\mathbf{e}_x$ and $\mathbf{e}_y$ bases vectors are, respectively, in the southerly and easterly direction.

We now calculate the apparent acceleration component $\mathbf{a}_c$ due to the Coriolis term in equation (8.31)

$$\begin{aligned}\mathbf{a}_c &\simeq -2\left\{\omega\left[-\cos(\lambda)\mathbf{e}_x + \sin(\lambda)\mathbf{e}_z\right]\times\left[-gt\mathbf{e}_z\right]\right\}\\ &\simeq -2\omega gt\cos(\lambda)\left[\mathbf{e}_x \times \mathbf{e}_z\right]\\ &\simeq 2\omega gt\cos(\lambda)\mathbf{e}_y.\end{aligned} \tag{8.34}$$

Inserting equation (8.34) in equation (8.31) we find the apparent acceleration of the particle as seen from the Earth's surface

$$\mathbf{a}_r \simeq 2\omega gt\cos(\lambda)\mathbf{e}_y - g\mathbf{e}_z. \tag{8.35}$$

If we assume that the initial conditions for the position of the particle are $x_0 = y_0 = 0$ and $z_0 = h$, we have after twice integrating equation (8.35)

$$\mathbf{r}(t) \simeq \frac{1}{3}\omega gt^3\cos(\lambda)\mathbf{e}_x + \left(h - \frac{1}{2}gt^2\right)\mathbf{e}_z. \tag{8.36}$$

When the particle reaches the Earth's surface we will have $t \simeq \sqrt{2h/g}$, and finally for the horizontal deviation

$$d \simeq \frac{1}{3}\omega\cos(\lambda)\sqrt{\frac{8h^3}{g}}. \tag{8.37}$$

Thus, if an object is dropped from a height of 100 m at latitude $45°$ north, it is deflected approximately by only 1.55 cm (we neglected any friction brought up by the presence of the atmosphere).

2. Foucault's pendulum. We set the origin of the noninertial xyz coordinate system at the equilibrium point of the pendulum and the z-axis along the local vertical. Describe the motion of the pendulum of length l and mass m in the small angle limit, taking into account the rotation of the Earth.

Solution. The equation of motion is

$$\mathbf{a}_r = \mathbf{g} + \frac{\mathbf{T}}{m} - 2\boldsymbol{\omega}\times\mathbf{v}_r, \tag{8.38}$$

where $\mathbf{T}$ is the tension in the pendulum. If we restrict ourselves to small oscillations, we can write

$$\mathbf{T} \simeq -T\frac{x}{l}\mathbf{e}_x - T\frac{y}{l}\mathbf{e}_y + T\mathbf{e}_z, \tag{8.39}$$

where we neglected second and higher order terms in x/l and y/l. As in the previous example, we write

$$\mathbf{g} = -g\mathbf{e}_z, \tag{8.40}$$

and

$$\begin{aligned} \omega_x &= -\omega\cos(\lambda) \\ \omega_y &= 0 \\ \omega_z &= \omega\sin(\lambda). \end{aligned} \tag{8.41}$$

Again limiting ourselves to small angular displacements, we can write for the velocity of the pendulum

$$\begin{aligned} (\mathbf{v}_r)_x &= \dot{x} \\ (\mathbf{v}_r)_y &= \dot{y} \\ (\mathbf{v}_r)_z &\simeq 0. \end{aligned} \tag{8.42}$$

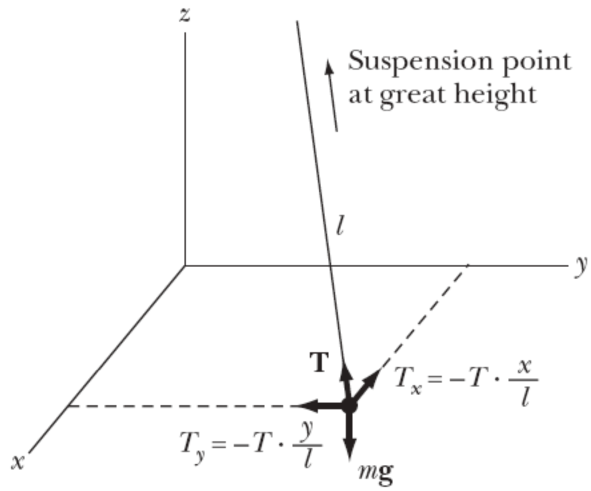

Figure 8-7 – Geometry of Foucault's pendulum. The acceleration vector is along the $-z$-axis, and the tension $\mathbf{T}$ is broken down into components along the x-, y-, and z-axes.

Using equations (8.41) and (8.42) to evaluate the Coriolis effect in equation (8.38), we can find the apparent acceleration of the pendulum as seen near the surface of the Earth (i.e., in the noninertial system) to be

$$\begin{aligned}\mathbf{a}_r \simeq & \left[-\frac{T}{m}\frac{x}{l} + 2\omega\dot{y}\sin(\lambda)\right]\mathbf{e}_x + \left[-\frac{T}{m}\frac{y}{l} - 2\omega\dot{x}\sin(\lambda)\right]\mathbf{e}_x \\ & + \left[\frac{T}{m} + 2\omega\dot{y}\cos(\lambda) - g\right]\mathbf{e}_z.\end{aligned} \tag{8.43}$$

If we concentrate on the motion in the xy plane, and make the following substitutions $T \simeq mg$, $\omega_0{}^2 \equiv T/ml \simeq g/l$, and $\omega_z = \omega\sin(\lambda)$ we find from equation (8.43)

$$\begin{aligned}\ddot{x} + \omega_0{}^2 x &\simeq 2\omega_z\dot{y} \\ \ddot{y} + \omega_0{}^2 y &\simeq -2\omega_z\dot{x},\end{aligned} \tag{8.44}$$

which is a system of two coupled second order differential equations. In order to facilitate the solution of the system, we multiply the second these equations by the unit imaginary number i and add it to the first equation. Then, defining the following complex variable

$$q \equiv x + iy, \tag{8.45}$$

we have from equations (8.44) that

$$\ddot{q} + 2i\omega_z\dot{q} + \omega_0{}^2 q \simeq 0. \tag{8.46}$$

As we saw in Chapter 2 on oscillations, equation (8.46) describes the motion of a damped oscillator (with the difference that the damping factor is, in this case, purely imaginary). Referring to the results obtained in the aforementioned chapter, we can write the solution to equation (8.46) to be

$$q(t) \simeq Ae^{-i\omega_z t} \cos\left(t\sqrt{\omega_z^{\ 2} + \omega_0^{\ 2}} - \delta\right). \tag{8.47}$$

We see that if the rotation of the Earth were ignored, we would retrieve the usual motion of a harmonic oscillator motion with

$$q(t) \simeq A\cos(\omega_0 t - \delta), \qquad \omega_z = 0, \tag{8.48}$$

and ω_0 is thus identified with the oscillation frequency of the pendulum. This frequency is much greater that the angular frequency of rotation of the Earth, which performs one complete rotation in approximately 24 hours. So, using the fact that $\omega_0 \gg \omega_z$ in equation (8.47) we have

$$\begin{aligned} q(t) &\simeq Ae^{-i\omega_z t} \cos(\omega_0 t - \delta) \\ &\simeq A\left[\cos(\omega_z t) - i\sin(\omega_z t)\right]\cos(\omega_0 t - \delta), \end{aligned} \tag{8.49}$$

which implies, using equation (8.45), that (assuming we chose the initial condition such that A is real)

$$\begin{aligned} x(t) &\simeq A\cos(\omega_z t)\cos(\omega_0 t - \delta) \\ y(t) &\simeq -A\sin(\omega_z t)\cos(\omega_0 t - \delta). \end{aligned} \tag{8.50}$$

It now becomes easy to see that as the pendulum is oscillating at a frequency ω_0, it also performs a **precession**, or rotation in the xy plane at a frequency of ω_z. The position angle made by the axis of oscillation in the xy plane will change with time as the pendulum rotates, and it is given by

$$\begin{aligned} \theta(t) &\simeq \tan^{-1}\left[\frac{y(t)}{x(t)}\right] \\ &\simeq \tan^{-1}\left[\frac{-\sin(\omega_z t)}{\cos(\omega_z t)}\right] \\ &\simeq -\omega_z t = -\omega t \sin(\lambda). \end{aligned} \tag{8.51}$$