Appendix: Orthogonal Curvilinear Coordinates

Notes:

We define the infinitesimal spatial displacement vector $d\mathbf{x}$ in a given orthogonal coordinate system with

$$d\mathbf{x} = dx^i e_i,$$  \hspace{1cm} (II.1)

where the Einstein summation convention was used, $dx^i$ is a contravariant component and $e_i$ is a basis vector ($i = 1, 2, 3$). The length interval $ds$ is thus given by

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x}$$
$$= (dx^i e_i \cdot dx^j e_j)$$
$$= (e_i \cdot e_j) dx^i dx^j$$
$$= g_{ij} dx^i dx^j = dx^i dx^i,$$  \hspace{1cm} (II.2)

where the orthogonality of the coordinate system is specified by $e_i \cdot e_j = g_{ij}$ with the metric tensor $g_{ij} = 0$ when $i \neq j$, and $dx_i$ is the covariant component. Please note that basis vectors are not unit vectors, i.e., $e_i \cdot e_i \neq 1$ in general. Equation (II.2) can be used to similarly define the inner product between any two vectors with

$$\mathbf{A} \cdot \mathbf{B} = g_{ij} A^i B^j$$
$$= A^i B_i.$$  \hspace{1cm} (II.3)

Since the covariant and contravariant components are generally different from one another in non-Cartesian coordinate systems, it is often more desirable to introduce a new set of so-called ordinary or physical components that preserve the inner product without explicitly bringing in both types of components or the metric tensor.

We start by rewriting equation (II.2) as

$$ds^2 = \left(h_1 du^1 \right)^2 + \left(h_2 du^2 \right)^2 + \left(h_3 du^3 \right)^2$$  \hspace{1cm} (II.4)
for the orthogonal coordinate system \((u^1,u^2,u^3)\). A comparison with equation (II.2) reveals that \(h_i^2 = g_{ii}\). For example, Cartesian coordinates have \(h_1 = h_2 = h_3 = 1\), cylindrical coordinates \((\rho, \theta, z)\) have \(h_1 = h_3 = 1\), \(h_2 = \rho\), and spherical coordinates \((r, \theta, \phi)\) have \(h_1 = 1\), \(h_2 = r\), and \(h_3 = r\sin(\theta)\). Going back to equation (II.3) for the inner product, we now define the physical coordinates \(\vec{A}_i\) of a vector \(\mathbf{A}\) such that

\[
\mathbf{A} \cdot \mathbf{B} = \vec{A}_i \vec{B}_i, \quad (\text{II.5})
\]

where the use of subscripts has no particular meaning (i.e., a subscript does not imply a covariant component). A comparison with equation (II.3) implies that the physical components are related to the covariant and contravariant components through

\[
\vec{A}_i = h_j A^j = h_i^{-1} A_i. \quad (\text{II.6})
\]

The first thing we should notice is that the physical components allow the use of a unit basis \(\hat{e}_i\) since

\[
\mathbf{e}_i \cdot \mathbf{e}_j = g_{ij} = h_i h_j \delta_{ij} = h_i h_j \left(\hat{e}_i \cdot \hat{e}_j\right) = \left(h_i \hat{e}_i\right) \cdot \left(h_j \hat{e}_j\right). \quad (\text{II.7})
\]

In fact, we could have alternatively justified the introduction of the physical components by the desire to use a unit basis with

\[
dx = h_1 du^1 \hat{e}_1 + h_2 du^2 \hat{e}_2 + h_3 du^3 \hat{e}_3 \quad (\text{II.8})
\]

or in general

\[
\mathbf{A} = \vec{A}_i \hat{e}_i + \vec{A}_2 \hat{e}_2 + \vec{A}_3 \hat{e}_3. \quad (\text{II.9})
\]

It should now be clear that what we usually specify as coordinates (e.g., \((\rho, \theta, z)\) and \((r, \theta, \phi)\)) correspond to the contravariant components of \(dx\), while the physical coordinates are those for which the components of \(dx\) have units of length (e.g., \((d\rho, \rho d\theta, dz)\) and \((dr, r d\theta, r \sin(\theta) d\phi)\)).

We now define the different differential operators using the physical coordinates, starting with the gradient. To do so, we first consider the differential of a scalar function \(f\)
\[ df = \frac{\partial f}{\partial u^i} \, du^i = \nabla f \cdot dx = \nabla f \cdot \left( \sum_i h_i du^i \hat{e}_i \right) = \sum_i h_i du^i \hat{e}_i, \]  

which from the first and last equations implies that

\[ \nabla f = \frac{1}{h_1} \frac{\partial f}{\partial u^1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u^2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u^3} \hat{e}_3. \]  

This leads to the following relations for the cylindrical and spherical coordinate systems

\[ \nabla f = \frac{\partial f}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \theta} \hat{e}_\theta + \frac{\partial f}{\partial \phi} \hat{e}_\phi, \]

\[ \nabla f = \frac{\partial f}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \phi} \hat{e}_\phi, \]  

respectively. For the divergence of a vector we consider an infinitesimal cube, as shown in Figure 1, and use the divergence theorem.

![Figure 1](image-url) - Infinitesimal volume of integration, where we do not differentiate between \( du^i \) and \( du_i \).
\[ \int_V \nabla \cdot \mathbf{A} d^3x = \nabla \cdot \mathbf{A} h_1 h_2 h_3 du^1 du^2 du^3 = \int_S \mathbf{A} \cdot \mathbf{n} da, \]  

(II.13)

which for a small enough cube we can write as

\[ \int_S \mathbf{A} \cdot \mathbf{n} da = \left[ \bar{A}_1 h_2 h_3 \right]_{\text{left}} \left[ du^2 du^3 \right] + \left[ \bar{A}_2 h_1 h_3 \right]_{\text{front}} \left[ du^1 du^3 \right] \\
+ \left[ \bar{A}_3 h_1 h_2 \right]_{\text{bottom}} \left[ du^1 du^2 \right] \]  

(II.14)

since in general \( h_i \) can vary across the dimensions of the cube. A comparison with equation (II.13) reveals that

\[ \nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u^1} (\bar{A}_1 h_2 h_3) + \frac{\partial}{\partial u^2} (\bar{A}_2 h_1 h_3) + \frac{\partial}{\partial u^3} (\bar{A}_3 h_1 h_2) \right]. \]  

(II.15)

We then respectively have for cylindrical and spherical coordinates

\[ \nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \bar{A}_r) + \frac{1}{\rho} \frac{\partial \bar{A}_\theta}{\partial \theta} + \frac{\partial \bar{A}_z}{\partial z} \]  

\[ \nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \bar{A}_r) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta} (\sin(\theta) \bar{A}_\theta) + \frac{1}{r \sin(\theta)} \frac{\partial \bar{A}_\phi}{\partial \phi}. \]  

(II.16)

The Laplacian is readily evaluated by setting \( \mathbf{A} = \nabla f \) and inserting equations (II.12) in equations (II.16). We then have the corresponding relations

\[ \nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \frac{\partial f}{\partial \rho}) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \]  

\[ \nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial f}{\partial r}) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial f}{\partial \theta} \right) + \frac{1}{r \sin^2(\theta)} \frac{\partial^2 f}{\partial \phi^2} \]  

(II.17)

for cylindrical and spherical coordinates, respectively.

Finally, for the curl we use Stokes’ Theorem using an infinitesimal surface as shown in Figure 2

\[ \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} da = (\nabla \times \mathbf{A}) \cdot \left( \alpha h_2 h_3 du^2 du^3 \mathbf{\hat{e}}_1 + \beta h_1 h_2 du^1 du^3 \mathbf{\hat{e}}_2 + \gamma h_1 h_2 du^1 du^2 \mathbf{\hat{e}}_3 \right) \]  

\[ = \oint_c \mathbf{A} \cdot d\mathbf{l}, \]  

(II.18)
Figure 2 – Infinitesimal loop of integration for the derivation of the curl, projection in the $(u', u'')$-plane.

where \( \mathbf{n} = \alpha \hat{e}_1 + \beta \hat{e}_2 + \gamma \hat{e}_3 \). For this infinitesimal loop we can consider the different projections on the three $(u', u'')$-planes and write (using the first two terms of the corresponding Taylor expansions)

\[
\oint_C \mathbf{A} \cdot d\mathbf{l} = \alpha \left[ \frac{1}{2} \left( \frac{\partial}{\partial u'} (\mathbf{A}_2 h_2) - \frac{\partial}{\partial u''} (\mathbf{A}_3 h_3) \right) \hat{e}_1 + \frac{1}{2} \left( \frac{\partial}{\partial u'} (\mathbf{A}_1 h_1) - \frac{\partial}{\partial u''} (\mathbf{A}_3 h_3) \right) \hat{e}_2 + \frac{1}{2} \left( \frac{\partial}{\partial u'} (\mathbf{A}_1 h_1) - \frac{\partial}{\partial u''} (\mathbf{A}_2 h_2) \right) \hat{e}_3 \right]
\]

Equating equations (II.18) and (II.19) we must have

\[
\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u'} (\mathbf{A}_1 h_1) - \frac{\partial}{\partial u''} (\mathbf{A}_2 h_2) \right] \hat{e}_1 + \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u'} (\mathbf{A}_2 h_2) - \frac{\partial}{\partial u''} (\mathbf{A}_3 h_3) \right] \hat{e}_2 + \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u'} (\mathbf{A}_3 h_3) - \frac{\partial}{\partial u''} (\mathbf{A}_1 h_1) \right] \hat{e}_3.
\]

We then respectively write for the cylindrical and spherical coordinate systems.
\[ \nabla \times \mathbf{A} = \left[ \frac{1}{\rho} \frac{\partial \tilde{A}_\rho}{\partial \theta} - \frac{\partial \tilde{A}_\theta}{\partial z} \right] \hat{\mathbf{e}}_\rho + \left[ \frac{\partial \tilde{A}_\theta}{\partial z} - \frac{\partial \tilde{A}_z}{\partial \rho} \right] \hat{\mathbf{e}}_\theta + \frac{1}{\rho} \left[ \frac{\partial}{\partial \rho} (\rho \tilde{A}_\theta) - \frac{\partial \tilde{A}_\rho}{\partial \theta} \right] \hat{\mathbf{e}}_z \]

\[ \nabla \times \mathbf{A} = \frac{1}{r \sin(\theta)} \left[ \frac{\partial}{\partial \theta} (\sin(\theta) \tilde{A}_\phi) - \frac{\partial \tilde{A}_\phi}{\partial \theta} \right] \hat{\mathbf{e}}_\phi + \frac{1}{r \sin(\theta)} \frac{\partial \tilde{A}_\rho}{\partial \phi} - \frac{1}{r} \frac{\partial \rho}{\partial r} (r \tilde{A}_\phi) \hat{\mathbf{e}}_\phi \quad (\text{II.21}) \]

\[ + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r \tilde{A}_\phi) - \frac{\partial \tilde{A}_r}{\partial \theta} \right] \hat{\mathbf{e}}_r. \]