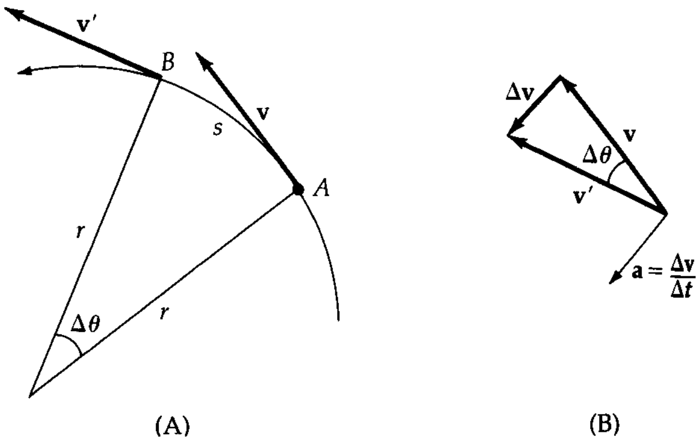
Uniform circular motion

Speed constant, but velocity changing.



Δv and **a** = **Δv/**Δt point to center.

$s = r\,\Delta\theta \Rightarrow \Delta\theta = s/r = v\,\Delta t / r$,

also $|\Delta v| = v\,\Delta\theta$ in small Δ limit

$\Rightarrow |\Delta v| = v^2\,\Delta t/r \Rightarrow |a| = v^2 / r$,

<u>centripetal acceleration</u>

Since **a** points to center of circle,

$\mathbf{F} = m\,\mathbf{a} = -\,m\,v^2 / r\ \hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector in the radial direction.

Now, since $v = 2\,\pi\,r / P$, and $P^2 = k\,r^3$ Newton realized that

$$F_{grav} = \frac{4p^2 m}{kr^2}$$, an <u>inverse square law</u>.

Newton's Law of Gravitation

Mutual attractive force between any two point masses m_1 and m_2 along the line joining them,

$$F_{grav} = \frac{G m_1 m_2}{r^2}.$$

It turns out that $G = 6.67 \times 10^{-11}$ m^3/(kg s^2).

A key extension of this result:

The gravitational force exerted by a spherical body is the same as if it's entire mass were concentrated at it's center. Therefore, at the Earth's surface,

$$F_{grav} = \frac{G M_E m}{R_E^{\,2}} = mg \;, \text{ where } g = GM_E/R_E^{\,2} = 9.80 \text{ m/s}^2$$

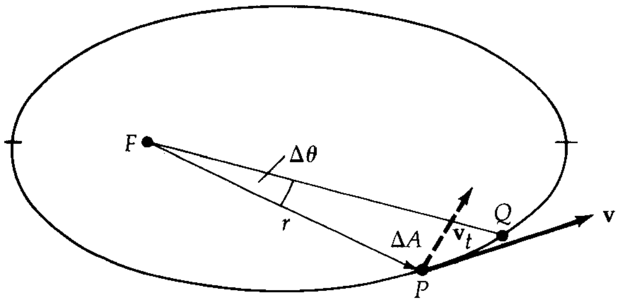
Kepler's Laws and Newtonian Mechanics

(1) Elliptical orbits

F = m **a** and $F = GMm/r^2$ => orbits are conic sections.

Kepler's 1st Law a special case.

(2) Law of Areas



$\Delta\theta = v_t\, \Delta t/r$ for small Δt

Also,

$\Delta A = 1/2\ r\ r\Delta\theta = 1/2\ r\ v_t\ \Delta t$

$\Rightarrow$ $dA/dt = 1/2\ r\ v_t$.

This quantity is constant according to Kepler's 2nd Law. Why?

Kepler's Laws and Newtonian Mechanics

Newtonian derivation of 2nd Law:

Angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,(\mathbf{r} \times \mathbf{v})$

Cross product: *magnitude* |r||p| sin θ, where θ is relative angle of **r**, **p**

direction by right hand rule

Torque d**L**/dt = $\mathbf{v} \times \mathbf{p} + \mathbf{r} \times d\mathbf{p}/dt = \mathbf{r} \times \mathbf{F} = 0$ if **F** is radial.

Therefore, $L = m\, r\, v \sin\theta = m\, r\, v_t = m\, H$ = constant, where $H = r\, v_t$.

$$\therefore \frac{dA}{dt} = \frac{1}{2} r v_t = \frac{H}{2} = \text{constant, proving 2nd Law.}$$

Can also write $$H = 2\frac{dA}{dt} = 2\frac{A}{P} = \frac{2pab}{P}.$$

Consequences:

At perihelion and aphelion, the velocity is transverse to radius vector.

$H = r\, v_t = r_p\, v_p = r_a\, v_a$

=>

$$v_p = \frac{2p\,ab}{P}\frac{1}{r_p} = \frac{2p\,ab}{P}\frac{1}{a(1-e)},$$

$$v_a = \frac{2p\,ab}{P}\frac{1}{r_a} = \frac{2p\,ab}{P}\frac{1}{a(1+e)}.$$

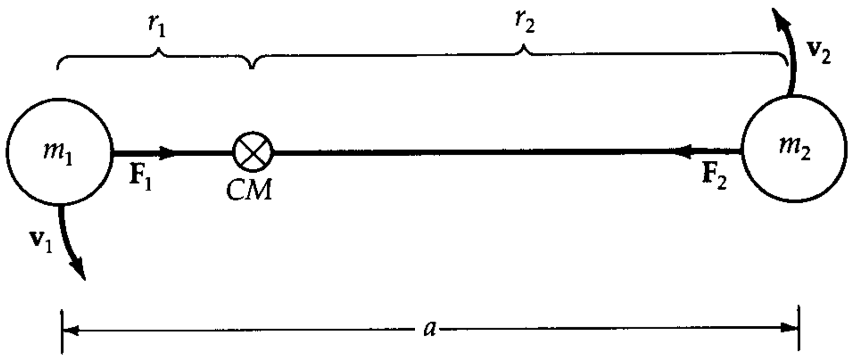
Use $b = a(1\text{-}e^2)^{1/2}$

=>

$$v_p = \frac{2p\,a}{P}\left[\frac{1+e}{1-e}\right]^{1/2},$$

$$v_a = \frac{2p\,a}{P}\left[\frac{1-e}{1+e}\right]^{1/2}.$$

(3) Newton's form of Kepler's 3rd Law



Two objects orbit around their stationary center of mass (CM) at distances r_1 and r_2.

CM stays fixed (or moves at constant velocity) if no external forces

Both objects orbit around CM with period P.

Combine $F_1 = m_1 v_1^2/r_1$, $F_2 = m_2 v_2^2/r_2$,

$P = 2\, p\, r_1/v_1 = 2\, p\, r_2/v_2$, and $F_1 = F_2$ to obtain

$m_1 r_1 = m_2 r_2$, which defines position of CM.

Now $a = r_1 + r_2$ $\Rightarrow r_1 = \dfrac{m_2 a}{m_1 + m_2}.$

Also, Newton's Law of Gravity yields

$$F_1 = F_2 = F_{grav} = \frac{G m_1 m_2}{a^2}$$

$$\therefore P^2 = \frac{4p^2 m_1 r_1}{F_1} \Rightarrow P^2 = \frac{4p^2 a^3}{G(m_1 + m_2)}.$$

More complete form of Kepler's 3rd Law - takes into account motion of both bodies.

Using Kepler's 3rd Law

Consider two objects in two separate orbits. Form ratios of previous equation:

$$(P/P')^2 = \left[(m_1' + m_2')/(m_1 + m_2)\right](a/a')^3.$$

For two objects orbiting the Sun,

$$m_1 + M_{Sun} \approx m_1' + M_{Sun} \approx M_{Sun}, \quad \text{so that}$$

$$(P/P')^2 = (a/a')^3, \quad \text{where } P' = 1 \text{ yr and } a' = 1 \text{ AU}.$$

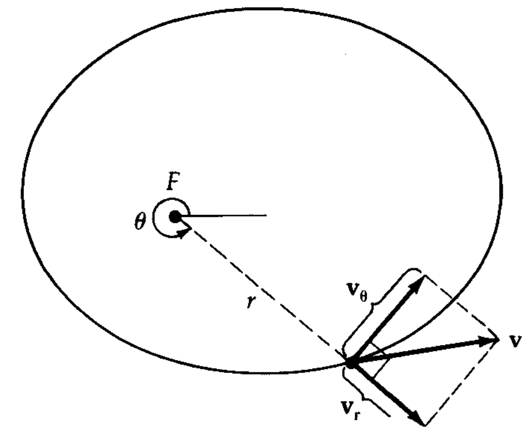
e.g., Mars, a = 1.52 AU =>

$P_{\text{Mars}} = (1.52)^{3/2}$ yr = 1.881 yr.

Orbital Velocity

Previous expressions for $H = r\, v_t$ yield

$$v_t \equiv v_q \equiv r\frac{dq}{dt} = \frac{2p}{P}\frac{a^2}{r}\left(1 - e^2\right)^{1/2}$$



Combine with polar equation for ellipse

$$r = \frac{a(1 - e^2)}{1 + e\cos q}$$ to obtain

$$v_r \equiv \frac{dr}{dt} = \frac{2pa}{P} e\sin q\left(1 - e^2\right)^{-1/2}$$

$$v_q \equiv r\left(\frac{dq}{dt}\right) = \frac{2pa}{P}(1 + e\cos q)\left(1 - e^2\right)^{-1/2}$$

The total orbital speed is then

$$v^2 = v_r^2 + v_\theta^2 = \left(\frac{2\pi a}{P}\right)^2 \left(\frac{1 + 2e\cos\theta + e^2}{1 - e^2}\right)$$

Eliminate cos θ using the polar equation of an ellipse to obtain

$$v^2 = G(m_1 + m_2)\left(\frac{2}{r} - \frac{1}{a}\right)$$

The vis viva equation.

This equation is actually a statement of energy conservation.

Conservation of Energy in a Two-Body System

Orbital speed used previously is the <u>relative</u> speed of the two bodies.

$v = v_1 + v_2$, where v_1 and v_2 are the individual speeds.

Also, $m_1 v_1 = m_2 v_2$ (Newton's 1st Law) which leads to

$$v = \frac{v_1(m_1 + m_2)}{m_2} = \frac{v_2(m_1 + m_2)}{m_1}.$$

Substituting into the vis viva equation yields

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{Gm_1 m_2}{r} = -\frac{Gm_1 m_2}{2a}.$$

1st two terms: Kinetic Energy (KE) of two objects around CM

3rd term: Potential Energy (PE) of the interaction

RHS: The total energy. It is conserved for an isolated system. Also, negative for a bound system.

Satellite orbits

At the Earth's surface, get escape speed from energy conservation:

$$\left(1/2\, mv^2\right) - \left(GM_\oplus m/R_\oplus\right) = 0,$$ i.e., the case where v drops to 0 at an infinite radius => $$v_{esc} = \left(2GM_\oplus/R_\oplus\right)^{1/2}.$$

The circular speed at the Earth's surface is

$$v_{c0} = \left(GM_\oplus/R_\oplus\right)^{1/2},$$ so that $$v_{esc} = \sqrt{2}v_{c0}.$$

In general, a satellite reaches its effective launch velocity far above the Earth's surface, when its fuel runs out. The subsequent orbit depends on the relation of its speed to the circular speed and escape speed <u>at that radius</u>.

Satellite orbits

Let $v_c = \left(GM_{\oplus}/r\right)^{1/2}$, where $r = R_{\oplus} + h$, h is the height above surface.

Then $v_{esc} = \sqrt{2}v_c$ at this radius.

If the burnout velocity is parallel to the Earth's surface, we get the following possible outcomes by studying the vis viva equation,

$$v/v_c = \left(2 - r/a\right)^{1/2} \Rightarrow a = \frac{r}{2 - \left(v/v_c\right)^2}:$$

$v < v_c$ elliptical orbit, start at apogee

$v = v_c$ circular orbit

$v_c < v < \sqrt{2}v_c$ elliptical orbit, start at perigee

$v = \sqrt{2}v_c$ $a = \infty$, parabola

$v > \sqrt{2}v_c$ hyperbola