Stability limits for the quasi-satellite orbit

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ABSTRACT

An asteroid moving around the Sun having approximately the same mean motion and mean longitude as a planet, but a different eccentricity, circles the planet like a retrograde satellite even when the distance is large enough so that it is not a bound satellite. If the orbits are coplanar, then the motion is stable in the secular approximation. When the orbits are inclined enough, an asteroid can be trapped into such a quasi-satellite (QS) motion for a finite period of time. The conditions under which this can occur are discussed, improved criteria for the recognition of this type of motion are developed, and numerical examples from real QS objects are provided.

Key words: celestial mechanics – minor planets, asteroids – planets and satellites: individual: Earth – Solar system: general.

1 INTRODUCTION

When an asteroid moves near a planet, like the Near Earth Asteroids (NEA) in the neighbourhood of the Earth, the orbit still remains essentially a heliocentric ellipse. However, the small perturbations due to the planet's presence at times nevertheless affect the motion considerably. This happens especially when there is a 1:1 resonance, that is, when the periods are (almost) the same. Well-known such cases are the Trojan and horseshoe orbits, while the quasi-satellite (hereafter QS) orbit is less well known, although the earliest publications on the subject is (to our knowledge) Jackson (1913). Also Henon & Guyot (1970) and Danielsson & Ip (1972) discussed this kind of orbital behaviour. Later such an orbit was used in astronautics Kogan (1990) which led Lidov & Vashkov'yak (1994a,b) to further investigate the subject. Also Mikkola & Innanen (1997) and Wiegert, Innanen & Mikkola (2000) (who were unaware of the earlier works in the subject) considered such orbits both analytically and numerically. Namouni (1999) considered the co-orbital motion of planets and asteroids in a rather general way, revealing that, for example, horseshoe orbits can change to QS orbits and back.

Later (Connors et al. 2002, 2004), a temporary QS of the Earth and also one of the planet Venus (Mikkola et al. 2004) were discovered. Since these real cases are only temporarily in a QS orbit, there is some motivation to improve theoretical considerations of the temporary trapping of a particle into QS-type motion. We will say that a QS is 'stable' if it cannot move away from the vicinity of

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the planet. However, when the orbital elements vary enough to allow the particle to change the type of motion (usually to horseshoe orbit) it has 'escaped'. Investigating the conditions for these transitions is the main subject of this paper.

The plan of the paper is as follows. First, we consider the behaviour of a QS in terms of the elliptic orbit approximation. This part is, in a sense, simple but helpful for a reader who is not an expert in this phenomenon. Then the approximate perturbative equations of motion are considered and finally a full secular theory is constructed for obtaining a firmness condition for the QS motion. This requires a numerical integration of a known function since analytical expressions cannot be constructed. With today's computers the evaluation of the integrals is, however, not much more time consuming than many of the functions that programmers take for granted.

2 CO-ORBITAL MOTIONS

There are three possible types of co-orbital motion of a small body associated with a planet: the well-known Trojan 'tadpole' orbits near the Lagrange points; horseshoe orbits along the planet's orbit; and QS motion remaining near the planet. Although regular co-orbital motion involves small perturbations which change the orbit, over one revolution the small body's motion is close to a Keplerian orbit. The longer-term motion, enforced by the perturbations in all three cases, shows its special nature in a frame co-rotating with the planet's revolution. The three different types of orbits are illustrated in Fig. 1. Here examples of the tadpole (TP) and horseshoe (HS) orbits were computed using a planet's mass of $m = 10^{-3}$ M$_\odot$ (to get wide enough orbits for illustration purposes), while for the QS
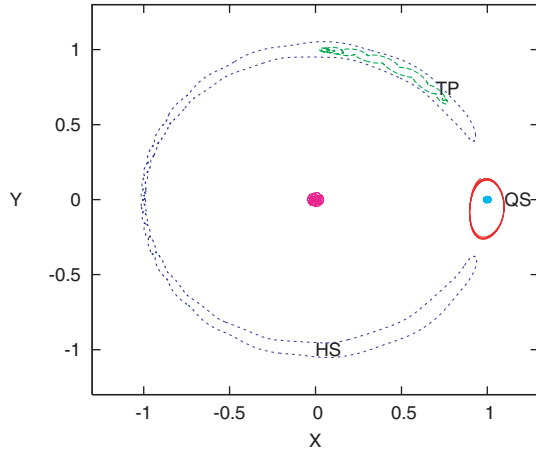
Figure 1. A demonstration of the three types of co-orbital motions. TP = tadpole (Trojan) motion, HS = horseshoe orbit and QS = quasi-satellite. These were obtained by numerical integration of the equations of motion. To get wide horseshoe and tadpole orbits, the mass value of $m = 10^{-3}$ was used.

a value $m = 3 \times 10^{-6}$ M$_\odot$, close to the mass of the Earth, was used. For simplicity the planet's orbit was taken as circular, of unit semimajor axis, and with the small body in its plane. Similarly, for the particle, initial values corresponding to a low eccentricity were given.

We consider the QS motion in the framework of the elliptic restricted three-body problem. This is justified by the fact that external perturbations typically affect the planet and the asteroid in much the same way so that the effect in their relative motion is not significant. Exceptions occur usually only due to close approaches to other planets.

Since the perturbations due to the planet change the orbital elements only slowly, a first approximation is simply the difference of two elliptic motions. This difference can then be used to construct an approximate secular theory of the motion. Due to the short distance between the planet and the QS, such a complete theory is difficult to construct and will remain approximate, and partly qualitative.

3 THEORY

We use the standard notations a, e, I, ω, Ω and M, for the semimajor axis, eccentricity, inclination, argument of perihelion, longitude of the ascending node and the mean anomaly, respectively. For the elements of the planet, the subscript 1, (e.g. M_1, e_1 and m_1 for the mean anomaly, eccentricity and the mass, respectively) is used. We further introduce

$$\alpha = \frac{a - a_1}{a_1}, \quad \varpi = \omega + \Omega, \tag{1}$$

the relative deviation of the semimajor axis from that of the planet and longitude of perihelion, and

$$\lambda = \varpi + M, \ \theta = \lambda - \lambda_1 \tag{2}$$

as the mean longitude and its difference from that of the planet.

It is convenient to use the planet's orbital plane as the xy-plane and take the direction of the pericentre of the planet's orbit as the x-axis direction, so that

$$\omega_1 = \Omega_1 = I_1 = 0, \lambda_1 = M_1.$$

This is the primary coordinate system used throughout this paper. Especially the xy-plane will be set into the plane of motion of the

planet. Because the orbital elements are often available in the ecliptic system, we give in Section 4.1 formulae for transforming orbital elements into the system used in this paper.

Assuming that e, e_1, $s = \sin(I/2)$ and θ are small quantities of the same order of magnitude (call it ϵ), then

$$\max(e, e_1, s, \theta) \sim \epsilon \ll 1. \tag{3}$$

A typical magnitude of $\epsilon \sim 0.1$ in the known examples. Since α must be much smaller than that in any co-orbital motion, we can assume $\alpha \ll \epsilon$ and, thus, α can often be neglected in comparison with the other small quantities. It is known that (Mikkola & Innanen 1997; Namouni 1999; Namouni, Christou & Murray 1999) in the stable QS motion $\alpha \sim \sqrt{m_1/\text{M}_\odot}$. Thus we can write

$$\sqrt{\frac{m_1}{\text{M}_\odot}} \overset{<}{\simeq} \alpha \ll \epsilon, \tag{4}$$

which can be used in the derivation of the approximate equations of motion (note that this relation is written in the squared form in the cited papers).

3.1 Geometry of the problem

To specialize the discussion to the QS state, we now look at some of its properties. We will proceed to expand the coordinates of the planet and of the QS to leading order in the small parameters (actually ϵ, defined in equation 3). For simplicity we assume

$$a = a_1 = 1$$

and re-introduce later the real scale of the orbit.

We use several coordinate systems: heliocentric and planetocentric, and rotating versions of both. The rotating coordinates rotate with the planet (thus not with constant rate) such that the x-axis goes through the Sun and the planet.

In Fig. 1, the QS-motion is illustrated in the simplest way: the planet's orbit is circular and the mutual inclination is zero. For the QS-orbit the realistic value of $e = 0.1$ was adopted for this illustration. In the co-rotating heliocentric frame

$$x = 1 - e\cos(M + \theta) + O(\epsilon^2) \tag{5}$$

$$y = 2e\sin(M + \theta) + O(\epsilon^2), \tag{6}$$

approximates this orbit quite well (but the illustration was obtained by numerical integration of the orbit).

In Fig. 2, the motion of a QS is illustrated in heliocentric coordinates. Here the full line represents the motion of the planet and short lines are drawn from the positions of the planet to those of the QS (the dots) at the corresponding moments of time. One notes that the asteroid stays on one side of the planet in inertial space. The direction is that of the aphelion of the asteroid's orbit [but see equation (8) for the effect of the planet's eccentricity].

Fig. 3 illustrates two QS orbits with different values of the phase-difference angle θ but in planetocentric (non-rotating) coordinates. One notes that the form and size of the relative orbit depend considerably on the angle θ but that the QS stays preferably on one side of the planet.

Finally, Fig. 4 illustrates the two (different θ) orbits in the rotating coordinate system. Here the trajectories look like two small ellipses shifted by the amount θ. The ellipses originate from the epicyclic motion (i.e. essentially from the eccentricity), which is not necessarily related to QS motion, but a QS motion is characterized by the libration of the angle θ (around zero) which causes the ellipse
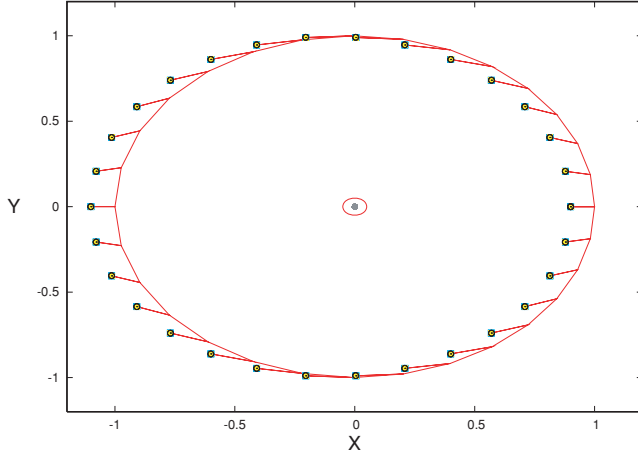
Figure 2. Motion of a QS in heliocentric coordinates. The full line illustrates the planet's motions. The dots represent the asteroid positions which are connected with short strokes to the planet's concurrent positions.
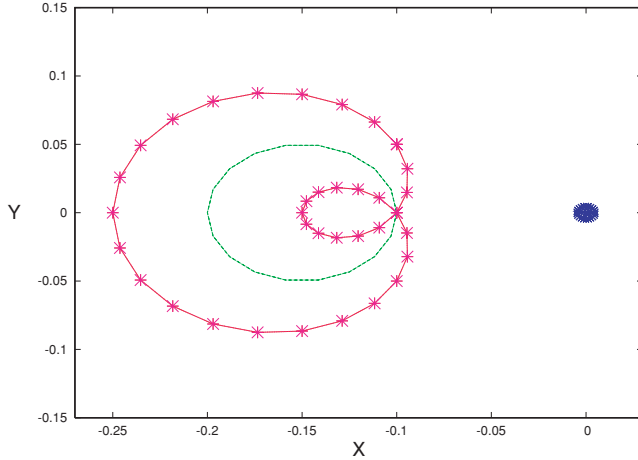

Figure 3. Motion of a QS in planetocentric coordinates. The big blue spot at origin is the planet. Two cases of QS trajectory are plotted: the green full line is for $\theta = 0$, and the red line with big starry dots is for $\theta = 0.05$ In both cases $e = 0.1$, $e_1 = 0$, $I = 0°$.

to move up and down (in this figure) in such a way that the planet remains inside the ellipse.

(i) *The planetocentric coordinates of the QS*, with the planet in its eccentric orbit as the guiding centre of epicyclic motion (Murray & Dermott 2000), are

$$
\Delta \approx [(3e_1 - e_1 \cos(2M_1) + e \cos(2M_1 - \varpi)
$$

$$
-3e \cos(\varpi) - 2\theta \sin(M_1))/2,
$$

$$
(2\theta \cos(M_1) - e_1 \sin(2M_1)
$$

$$
+e(\sin(2M_1 - \varpi) - 3\sin(\varpi)))/2,
$$

$$
2s \sin(M_1 - \Omega)], \tag{7}
$$

where $s = \sin(I/2)$ is half the sine of the mutual inclination of the orbits.
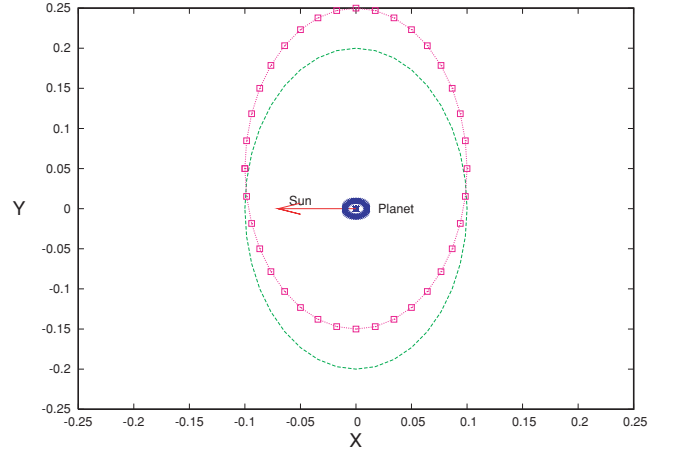

Figure 4. The same two orbits as in Fig. 3 but in the rotating coordinate system. These are essentially ellipses with axis ratio 1:2. In the long term, the centre of the ellipse oscillates back and forth (when θ librates around zero), but in such a way that the planet remains inside the ellipse (in QS state of motion). The colour coding is the same as in the previous figure.

(ii) *The mean value of the planetocentric vector* of the QS is

$$
\langle\Delta\rangle = -\frac{3}{2}(e - e_1) \approx -\frac{3}{2}\begin{pmatrix} e\cos(\varpi) - e_1 \\ e\sin(\varpi) \\ O(\epsilon^2) \end{pmatrix}. \tag{8}
$$

The vectors e and e_1 [here $e_1 = (e_1, 0, 0)$] are the eccentricity vectors (or the Runge–Lenz vectors) of the asteroid and the planet, respectively. Note that in any elliptic Kepler motion the position vector average is

$$
\langle r\rangle = -\frac{3}{2}ae_{|a=1} = -\frac{3}{2}e
$$

from which the above follows.

(iii) *In the rotating coordinate system*, the planetocentric coordinates are

$$
\Delta_r \approx \begin{pmatrix} e_1\cos(M_1) - e\cos(M_1 - \omega - \Omega) \\ \theta - 2e_1\sin(M_1) + 2e\sin(M_1 - \omega - \Omega) \\ 2s\sin(M_1 - \Omega) \end{pmatrix}. \tag{9}
$$

(iv) *The asteroid crosses the orbital plane of the planet* when the z-component of Δ_r is zero. In our approximation $M_1 = \Omega$ at this crossing of the planet's orbital plane and the coordinates of this point are

$$
\Delta_c \approx \begin{pmatrix} -e\cos(\omega) + e_1\cos(\Omega) \\ \theta - 2e\sin(\omega) - 2e_1\sin(\Omega) \\ 0 \end{pmatrix} \tag{10}
$$

or if $M_1 = \Omega + \pi$

$$
\Delta_c \approx \begin{pmatrix} e\cos(\omega) - e_1\cos(\Omega) \\ \theta + 2e\sin(\omega) + 2e_1\sin(\Omega) \\ 0 \end{pmatrix} \tag{11}
$$

which turn out to have an important role in the stability of a QS (see later).

3.2 Equations of motion

The secular perturbing function in the heliocentric coordinates is

$$R = m_1 \left\langle \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_1|} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}_1}{r_1^3} \right\rangle, \tag{12}$$

where $\langle \cdot \rangle$ means time average. The use of this is justified if the orbital elements do not change considerably during one period of the planet. This is the case if the particle does not come too close to the planet.

The usual Lagrange equations for the adopted elements $\alpha = (a - a_1)/a_1$, e, i, ϖ, Ω and θ can be written in the form (with $V = \sqrt{1 - e^2}$)

$$\dot{\alpha} = \frac{2}{na^2} \frac{\partial R}{\partial \theta},$$

$$\dot{\theta} = n - n_1,$$

$$-\frac{2}{na^2} \frac{\partial R}{\partial \alpha} + \frac{V - V^2}{na^2 e} \frac{\partial R}{\partial e} + \frac{1 - \cos(I)}{na^2 V \sin(I)} \frac{\partial R}{\partial I},$$

$$\dot{e} = \frac{V^2 - V}{na^2 e} \frac{\partial R}{\partial \theta} - \frac{V}{na^2 e} \frac{\partial R}{\partial \varpi},$$

$$\dot{I} = \frac{\cos(I) - 1}{na^2 V \sin(I)} \frac{\partial R}{\partial \theta} - \frac{1}{na^2 V \sin(I)} \frac{\partial R}{\partial \Omega},$$

$$\dot{\varpi} = \frac{V}{na^2 e} \frac{\partial R}{\partial e} + \frac{1 - \cos(I)}{na^2 V \sin(I)} \frac{\partial R}{\partial I},$$

$$\dot{\Omega} = \frac{1}{na^2 V \sin(I)} \frac{\partial R}{\partial I}. \tag{13}$$

As α and m_1 are small, the main term in the equation for $\dot{\theta}$ is

$$\dot{\theta} = n - n_1 \approx -\frac{3}{2} n_1 \alpha \tag{14}$$

and thus

$$\ddot{\theta} = -\frac{3}{a_1^2} \frac{\partial R}{\partial \theta}, \tag{15}$$

which integrates to

$$\frac{1}{2} \dot{\theta}^2 = \frac{3}{a_1^2} [R_x - R(\theta)], \tag{16}$$

where R_x is an integration constant. In this the time dependence of all the elements, other than θ has been neglected. This is justified because θ is by far the fastest variable (Mikkola & Innanen 1997; Namouni 1999) and the dependence of R on α can be neglected in a first approximation.

A more detailed demonstration of this is as follows. The time derivative of R can be written as

$$\dot{R} = \frac{\partial R}{\partial \theta} \dot{\theta} + \sum_{q \neq \theta} \frac{\partial R}{\partial q} \dot{q}, \tag{17}$$

where q has been used as a generic notation for the orbital elements. From equations (13) we see that

$$\dot{\theta} \propto \alpha \sim \sqrt{\frac{m_1}{M_\odot}}, \text{ but } \dot{q} = O(m_1) \text{ for } q \neq \theta.$$

Consequently, in the expression for $\dot{R}$, the latter terms are smaller than the θ-term by a factor of the order of $\sqrt{m_1}$ which is very small (for terrestrial planets at least). Thus the approximation $\dot{R} = \frac{\partial R}{\partial \theta} \dot{\theta}$ is justified.
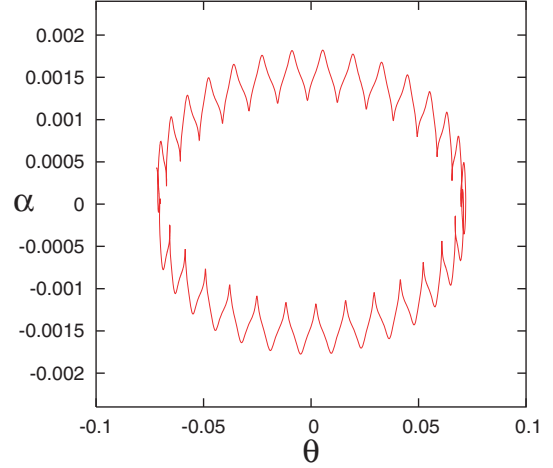

Figure 5. The behaviour of α as function of θ in stable QS motion. In this illustration the planet's mass was $m = 3 \times 10^{-6}$ and $e_1 = 0$, $e = 0.1$, $I = 0$. Data were obtained by numerical integration of the full equations of motion. The secular equations would describe only the average motion, that is, the short-period fluctuations that are clear here would not appear in the solution of the secular equations discussed in the text.

3.3 QS-motion

In Fig. 5, the motion of the relative change of semimajor axis α versus the angle θ is illustrated over one period of θ motion. The results are from direct numerical integration of the restricted three-body-problem. Thus, the short-period term (of period = one planetary period) is present and clearly visible in the figure. The secular equations (14) and (15) give this motion without the short period part. As will be seen in the next sections, a first approximation for the secular motion is a (nearly) harmonic oscillation of the variables θ and α. Note that this motion of θ corresponds to back and forth oscillations of the centre of the ellipse in Fig. 4. This, not the epicyclic motion, is the real defining point of QS motion since the epicyclic motion would take place even without any gravitational pull from the planet (Brasser et al. 2004).

However, the (α, θ) figure was obtained for zero inclination, in which case a stable motion is possible, while high enough inclination produces circumstances in which a particle can enter the QS-state of motion, stay there for some time and finally exit that state of motion.

3.4 Approximation by expansion

The secular perturbing function can be approximated by

$$R = \frac{m_1}{a_1} \left\langle \frac{1}{\Delta} \right\rangle, \tag{18}$$

because the indirect term is negligible in comparison with the direct one.

Using the above expressions for components of $\boldsymbol{\Delta}$ in (10) and, following Mikkola & Innanen (1997), we define σ, ψ and w by writing

$$\sigma \cos(\psi) = e \cos(\varpi) - e_1, \tag{19}$$

$$\sigma \sin(\psi) = e \sin(\varpi), \tag{20}$$

and

$$w = M_1 - \psi. \tag{21}$$

This leads to the expression

$$\Delta^2 = \sigma^2[1 + 3\sin^2(w)] + 4s^2\sin^2(\Psi + w)$$
$$+ 4\sigma\theta\sin(w) + \theta^2, \tag{22}$$

where

$$\Psi = \psi - \Omega. \tag{23}$$

After this the averaging over time is replaced by averaging over the angle w.

By (19) and (20) we have $\sigma = e^2 - 2ee_1\cos(\varpi) + e_1^2$, but it is more natural to use

$$\sigma = |\boldsymbol{e} - \boldsymbol{e}_1|, \tag{24}$$

which is the same as the previous expression to the order of the theory. The meaning of the angles ψ and Ψ can be found by considering their trigonometric functions: one notes that ψ gives the direction of the mean planetocentric position vector (see equation 8) and for Ψ one obtains

$$\sin(\Psi) = \frac{e\sin(\omega) + e_1\sin(\Omega)}{\sigma} \tag{25}$$

and

$$\cos(\Psi) = \frac{e\cos(\omega) - e_1\cos(\Omega)}{\sigma}, \tag{26}$$

which have a connection with the point at which the QS crosses the orbital plane of the planet (see equation 8). Thus we have found for each quantity in the expression for Δ^2 a clear 'physical' meaning that is independent of the coordinate system. Unfortunately, evaluation of the angle Ψ still requires complicated computations since the angles ω and Ω must first be obtained in the planet's system of coordinates (Section 4.1).

The problem is now to evaluate the average of $1/\Delta$ over the angle w. To further simplify the notation we set

$$\frac{1}{\Delta} = \frac{1}{\sigma}U, \tag{27}$$

where

$$U = \frac{1}{\sqrt{1 + 3\sin^2(w) + 4S^2\sin^2(\Psi + w) + 4T\sin(w) + T^2}},$$

and $S = s/\sigma$, $T = \theta/\sigma$.

Since the expression under the square root is a quadratic polynomial of T, we can use the well-known expansion in terms of Legendre polynomials:

$$\langle U \rangle = \sum_{n=0}^{\infty} \left\langle \frac{1}{\Delta_0^{2n+1}} P_{2n}\left(\frac{2\sin(w)}{\Delta_0}\right)\right\rangle T^{2n}, \tag{28}$$

where

$$\Delta_0^2 = 1 + 3\sin^2(w) + 4S^2\sin^2(\Psi + w). \tag{29}$$

and we have used the fact that all odd-order terms average to zero. Once the parameters S and Ψ are numerically known it is easy to obtain the expansion coefficients by numerical averaging over w.

3.5 Near-planar case

Analytically the averaging does not seem possible in a simple way. However, following Mikkola & Innanen (1997), a first approximation valid for $S \ll 1$, $T \ll 1$ can be obtained. With these (very restrictive) assumptions, U can be expanded in powers of S and T (considering them to be quantities of the same order of magnitude). Averaging the resulting expansion is easy by numerical integration. To improve the convergence, it seems natural to convert the result

back to the form $1/\sqrt{\text{series}}$... Since we are mainly interested in the stability of motion, a further expansion to second order in T gives

$$\langle U \rangle = a_0\left(\frac{1}{\Delta_e} + \frac{a_2}{\Delta_e^3}T^2 + ..\right) \tag{30}$$

where

$$a_0 = 0.686\,440\,250\,309\,176$$
$$\Delta_e^2 = 1 + S^2[1.123\,16 - 0.5386\cos(2\,\Psi)]$$
$$+ S^4[-0.676\,025 + 0.692\,779\cos(2\,\Psi)$$
$$- 0.109\,125\,\cos(4\,\Psi)] \tag{31}$$

and

$$a_2 = 0.073\,07 - [0.1423 + 0.2347\cos(2\Psi)]S^2. \tag{32}$$

Here we have carried out the original expansion to fourth order (taking S and T to be small quantities of the same order). This is an improvement to Mikkola & Innanen (1997) where all the expansions were carried only to second order and the result did not suggest a region of instability at all (and was thus valid only for very small inclinations).

3.6 Possibility of QS motion

Because of equation (15), one can now find regions of *possible stability* and *guaranteed instability* (i.e. possibility/impossibility of long-lasting QS-type motion), by considering the sign of the second-order term in the expansion for $\langle U \rangle$, that is, the sign of a_2. A more accurate version is obtained by numerically averaging the second derivative of U by

$$\langle U_{TT} \rangle = \left\langle -\Delta_0^{-3} + \frac{12\sin^2(w)}{\Delta_0^5}\right\rangle. \tag{33}$$

where Δ_0^2 is defined by (29).

In Fig. 6, the results obtained from the expansion coefficient a_2 are compared with the more accurate expression (33). One sees that the above series approximation gives qualitatively correct results, but quantitatively there is a clear discrepancy. One notes that the instability is guaranteed if inclination is large enough ($S \geqslant 0.5$ i.e. $I \sim \sigma$) and Ψ is near 0 or π while stability is possible for I very small and/or Ψ is near $\pi/2$.

In Fig. 7, the secular (effective) potential $\langle U \rangle$ is plotted as a function of T for selected values of $S = \sin(I/2)/\sigma$ at $\Psi = 0$. For $S = 0$
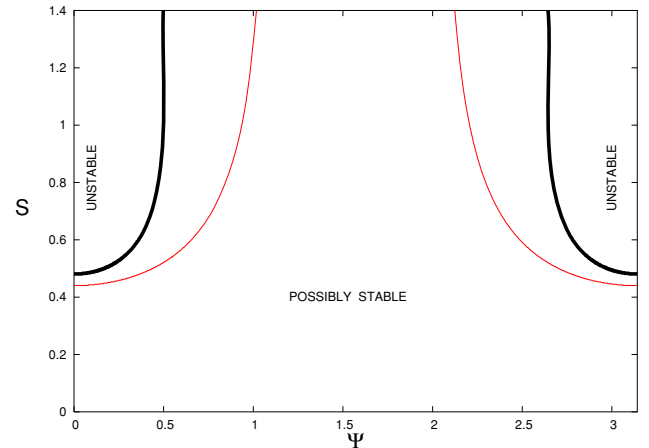

Figure 6. Limit of guaranteed instability (i.e. zero of the second derivative of $\langle U \rangle$ with respect to θ) as obtained by numerical integration of (33) (thick black line) and by the series approximation (32) (thin red line).
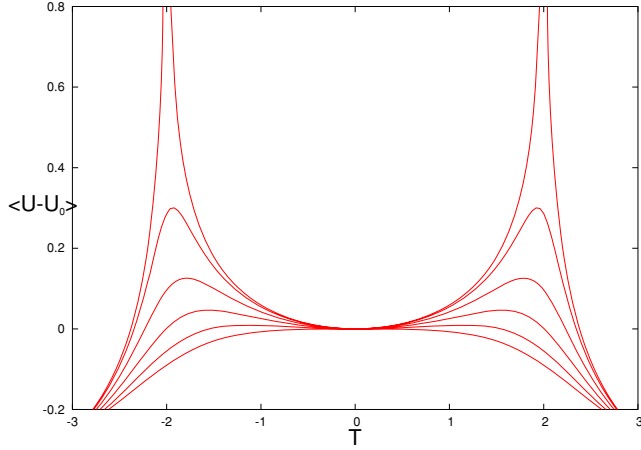
Figure 7. Behaviour of $\langle U(T) \rangle - \langle U(0) \rangle$ for different values of ($S = 0, 0.1$, 0.2, 0.3, 0.4 and 0.5). The uppermost curve (for $S = 0$) has a logarithmic infinity at $T = 2$, suggesting persistent stability, while the lowermost (i.e. $S = 0.5$ curve) is entirely convex and thus guarantees instability. The curves are given for $\Psi = 0$.

there is an infinite wall at $T = 2$. This corresponds, mathematically, to a collision of the planet and the QS (which cannot occur in our theory). For larger values of S, the maxima of the potential are finite but at $T = 0$ (i.e. $\theta = 0$) there is a minimum which makes stable oscillation possible. Finally, near $S = 0.5$ the potential becomes completely convex and stable motion is not possible.

If the angle $\Psi \sim \pi/2$, however, then all the curves resemble the plotted curve for $S = 0$, that is, there is the infinite potential wall. This is because $\Psi = \pi/2$ means that the crossing points are just ahead of and behind the planet in the orbit and, mathematically, a collision is possible. This is thus a stable situation (the region in the middle of Fig. 6).

A near-planet asteroid that is not (yet) in a QS-state of motion can enter that status when Ψ is in the unstable region by precession of Ψ to the stable region. The destabilization is the reverse process: the angle Ψ keeps precessing until it enters again the unstable region. However, the stability does not depend only on the angle Ψ, if the θ motion has too much energy, that is, $\dot{\theta} \approx -\frac{3}{2} n_1 (a - a_1)/a$ is too large, then the quantity R_x in (16) is so large that the 'θ-energy' is higher than the maximum of the secular potential $R(\theta)$ and an entry or exit can happen. A capture, for shorter or longer time, happens when an entry occurs and then Ψ precesses such that the secular potential $R(\theta)$ maxima become too high for the asteroid to escape. Such a capture can be quite brief, as in the case of 2003 NY107, or fairly lengthy as for 2004 GU9 (see Section 5).

The importance of the angle Ψ in the stability and the fact that the crossing-point coordinates in the orbital plane of the planet are [by (10) and (8)]

$$x_c = \pm \sigma \cos(\Psi) \tag{34}$$

$$y_c = \theta \pm 2\sigma \sin(\Psi), \tag{35}$$

leads us to a *physical interpretation of the stability:* consider the system in the spirit of averaging theory, from the point of view of the QS, replace the attracting planet by a mass ring, the points of which are located at the head of the vector x_r, y_r, z_r; when the angle $\Psi \sim \pi/2$ the crossing points are located in front of and just behind the planet in its orbit around the Sun. In this situation, the QS cannot move out of the neighbourhood of the planet without a close approach. If, on the other hand, $\Psi = 0$, the crossing points are

in inferior and superior conjunction. In this case the QS can move 'through' the ring (provided the inclination is large enough) and escape from the vicinity of the planet.

3.7 Full secular theory of θ oscillation

The full secular perturbing function is

$$R = \frac{m_1}{2\pi} \int_0^{2\pi} \left(\frac{1}{|r - r_1|} - \frac{r \cdot r_1}{r_1^3} \right) dM_1. \tag{36}$$

When the orbital elements for both the planet and the QS are given, one can evaluate the above integral numerically. One obtains it, as a function of θ, using for the mean anomalies the relation $M = M_1 - \varpi + \theta$. Then, by

$$\frac{1}{2}\dot{\theta}^2 + \frac{3}{a_1^2} R(\theta) = \frac{3}{a_1^2} R_x, \tag{37}$$

where R_x is an integration constant to be evaluated from the initial values, one obtains the period of θ-oscillation

$$P = 2 \int_{\theta_{min}}^{\theta_{max}} \frac{d\theta}{\sqrt{(6/a_1^2)[R_x - R(\theta)]}}. \tag{38}$$

Here the integration limits $\theta_{min}, \theta_{max}$ must be obtained by solving the equation

$$\frac{3}{a_1^2} R(\theta_{max}) = \frac{1}{2}\dot{\theta}_0^2 + \frac{3}{a_1^2} R(\theta_0), \tag{39}$$

where $\dot{\theta}_0 = n - n_1$ and all the necessary quantities are evaluated at the starting epoch. The value of $\theta_{min} \approx -\theta_{max}$ must be obtained similarly. Note that the function $R(\theta)$ is not exactly symmetrical (although it is so in the approximate theory) and thus both ends must be obtained separately.

For the QS to be stable, it is important that the quantities θ_{min} and θ_{max} exist. If these boundaries cannot be found (essentially in the region $T < 2$, that is, $\theta < |e - e_1|$), then the 'θ-energy' is at a higher level than the maximum of the effective potential. In other words, the difference of the semimajor axes is too large and there is no stable motion.

4 CRITERIA FOR QUASI-SATELLITE MOTION

Here we first describe some of the necessary computational tools and then give the conditions under which an asteroid may be considered to be in a QS state of motion.

4.1 Orbital angles in the planet's system

To obtain the orbital elements needed in the theory, one must convert to the system in which the *xy*-plane is the orbital plane of the planet and the *x*-axis is directed to the perihelion of the planet. If one deals with Earth QSs, this is particularly easy as the inclination of the Earth is very small: one simply rotates around the *z*-axis by the amount $-\varpi_1$. Considering the orbital angles

$I, \omega,$ and $\Omega,$

the only modification is to replace

$\Omega \to \Omega - \varpi_1$

(note that $\varpi = \omega + \Omega$ must be computed only *after* this operation, that is, ϖ is replaced by $\varpi - \varpi_1$).

More generally (and more accurately) one must compute the direction vectors P, Q and W and carry out the rotation properly by the following formulae.

First, one evaluates the P, Q, W-vectors as

$$P = \begin{pmatrix} +\cos(\omega)\cos(\Omega) - \sin(\omega)\sin(\Omega)\cos(I) \\ +\cos(\omega)\sin(\Omega) + \sin(\omega)\cos(\Omega)\cos(I) \\ +\sin(\omega)\sin(I) \end{pmatrix},$$

$$Q = \begin{pmatrix} -\sin(\omega)\cos(\Omega) - \cos(\omega)\sin(\Omega)\cos(I) \\ -\sin(\omega)\sin(\Omega) + \cos(\omega)\cos(\Omega)\cos(I) \\ +\cos(\omega)\sin(I) \end{pmatrix} \quad (40)$$

and

$$W = \begin{pmatrix} +\sin(\Omega)\sin(I) \\ -\cos(\Omega)\sin(I) \\ +\cos(I) \end{pmatrix},$$

for both the planet and the asteroid. To obtain the angles i, ω and Ω in the said system, one uses

$$P_z = W_p \cdot P_a = +\sin(\omega)\sin(I)$$
$$Q_z = W_p \cdot Q_a = +\cos(\omega)\sin(I)$$
$$W_x = P_p \cdot W_a = +\sin(\Omega)\sin(I)$$
$$W_y = Q_p \cdot W_a = -\cos(\Omega)\sin(I)$$
$$W_z = W_p \cdot W_a = +\cos(I),$$

where the subscripts 'p' and 'a' indicate the planet and the asteroid, respectively. Consequently the angles in the planet's system become

$$\omega = \text{atan}(P_z/Q_z)$$
$$\Omega = \text{atan}(-W_x/W_y)$$

and

$$I = \text{atan}\left(\sqrt{W_x^2 + W_y^2}/W_z\right).$$

4.2 Period evaluation

It is not possible to do the integration in (38) analytically, so that numerical evaluation is the only option.

First, compute the P, Q and W vectors in the planet's system according to the above. Then, in this system, the asteroid's coordinates are given by

$$r = a[P(\cos(E) - e) + \sqrt{1 - e^2}\, Q \sin(E)] \quad (41)$$

and those of the planet

$$x_p = a_1[\cos(E_1) - e_1], \quad y_p = a_1\sqrt{1 - e_1^2}\sin(E_1), \quad z_p = 0. \quad (42)$$

In the integration of

$$R = \frac{m_1}{2\pi} \int_0^{2\pi} \left(\frac{1}{|r - r_1|} - \frac{r \cdot r_1}{r_1^3} \right) dM_1, \quad (43)$$

one gives a value for the mean anomaly M_1 of the planet, obtains that of the QS by

$$M = M_1 - \varpi + \theta, \quad (44)$$

solves Kepler's equation, computes the coordinates (using $a = a_1$) and obtains the integrand. The best numerical method for evaluating this integral is probably the midpoint or trapezoidal rule because the integrand is periodic.

Obtaining the period is not completely trivial since the integrand in (38) has singularities at the ends of the integration interval. However, the transformation

$$\theta = \frac{(\theta_{\max} - \theta_{\min})}{2}\cos(\chi) + \frac{(\theta_{\max} + \theta_{\min})}{2},$$

eliminates those square-root singularities in the period integrand and gives

$$P = (\theta_{\max} - \theta_{\min}) \int_0^\pi \frac{\sin(\chi)d\chi}{\sqrt{(6/a_1^2)[R_x - R(\theta(\chi))]}}. \quad (45)$$

Note that if $R(\theta)$ is purely quadratic in θ, then the integrand above is constant. Thus, in general, the integrand changes only slowly and the numerical integration is easy.

4.3 Testing QS-state of motion

For oscillatory motion to be possible in the secular θ-potential $R(\theta)$, the equation

$$R(\theta) = R(\theta_0) + \frac{a_1^2}{6}\dot{\theta}_0^2$$

must have two solutions $\theta_{\min}$ and $\theta_{\max}$ (which satisfy $\theta_{\min} \approx -\theta_{\max}$). Here $\dot{\theta}_0 = n - n_1 \approx -\frac{3}{2}n_1(a - a_1)/a_1$ computed at the epoch. From the theory (mainly from Fig. 7) we see that the solutions $\theta_{\min}$ and $\theta_{\max}$ should be in the interval $|\theta| < 2\,|e - e_1|$, which condition must be satisfied initially as well. Note that the above conditions may be satisfied for a convex potential, but then there is no real period for then the quantity under the square root is negative.

The 'algorithm' for a first look at whether an asteroid is a QS may be summarized as follows.

(i) Rotate the orbital angles to the system in which the planet's orbital plane is the xy-plane. Normally, this can be done (to sufficient accuracy) by evaluating

$$\Omega_p = \Omega - \varpi_1$$

and computing new

$$\varpi_{\text{new}} = \omega + \Omega_{\text{new}}.$$

(ii) Compute the mean longitude difference at epoch

$$\theta_0 = M + \varpi_{\text{new}} - M_1$$

and the relative eccentricity σ from

$$\sigma^2 = |e - e_1| \approx e^2 + e_1^2 - 2ee_1\cos(\varpi_{\text{new}}).$$

(iii) Test if

$$|\theta_0| < 2\sigma.$$

If the answer is yes, a QS-state is possible.

(iv) Take for $\dot{\theta}$ the value

$$\dot{\theta}_0 = n - n_1 \approx -\frac{3}{2}n_1\frac{a - a_1}{a_1}$$

and obtain the constant R_x from

$$R_x = R(\theta_0) + \frac{a_1^2}{6}\dot{\theta}_0^2.$$

(v) Tabulate $R(\theta)$ for, say, $-3 < \theta < 3\sigma$ and plot the result together with the value of R_x to check if stable motion is possible. This can be done using the algorithm described in the beginning of Section 4.2.
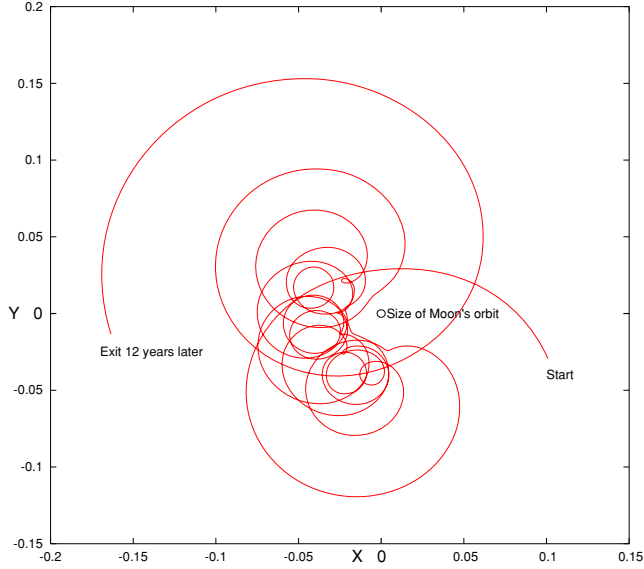
Figure 8. Geocentric motion of 2003 YN107, which is a short-term visitor in an Earth-QS orbit. The evolution of the orbit is mainly due to the change of the mean longitude difference θ. This asteroid is a transient QS, making only one θ-oscillation in the QS-state of motion.

5 EXAMPLES

There are some known asteroids that are now or have been/will be in QS-orbit about the Earth. Such examples are Cruithne (Wiegert, Innanen & Mikkola 1997), 2002 AA29 which is at present in a horseshoe orbit, but will enter a QS orbit in the future and 2003 YN107 which is at present in a transient QS orbit (Connors et al. 2002, 2004). A most interesting case is 2004 GU9 which is at present in QS orbit, near the middle of its near 1000 yr stay in QS trajectory.

Fig. 8 illustrates the numerically integrated motion of 2003 YN107 in geocentric non-rotating coordinates. One notes that the asteroid stays preferably to one side of the Earth in accordance with the theory. However, this asteroid is a transient QS only: the QS-state involves only one θ-oscillation period. It also comes so close to the Earth that the eccentricity changes considerably during this period, thus making the usefulness of the theory questionable.

In Fig. 9, the evolution of the semimajor axis of 2004 GU9 is plotted as a function of time over a period of 3000 yr centred at the present epoch. It is clear that this body used to be in a horseshoe orbit and will again exit to such an orbit, while in between it stays near the Earth in a QS state of motion.

5.1 A numerical example

For 2004 GU9, we obtain

$$\Psi = -1.438\,826\,08,\ \sigma = 0.150\,253\,787,\ S = 0.790\,699\,503,$$

[using Keplerian elements at epoch 53200 (MJD)]. The expansion

$$R \approx 0.000\,010\,4744 + 0.000\,029\,3191\,\theta^2 + 0.000\,180\,41\,\theta^4$$
$$+ 0.001\,348\,77\,\theta^6 + 0.010\,9462\,\theta^8, \tag{46}$$

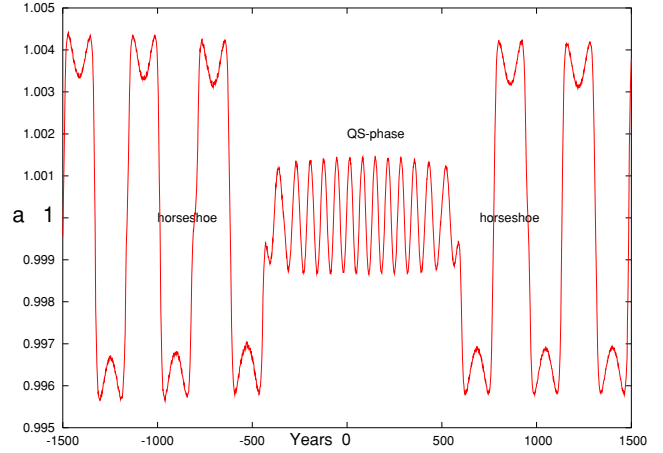then follows by numerically evaluating the coefficients in the expansion (28).

Figure 9. Behaviour of the semimajor axis of 2004 GU9, which stays as an Earth-QS for about a thousand years. At the present epoch it is near the halfway point of its stay near the Earth.

By (45) one can use this expansion to compute the θ-oscillation periods in different approximations

Approximations	Years
P_2	$= 75.5$
P_4	$= 69.1$
P_6	$= 68.1$
P_8	$= 67.9$
P_{a_2}	$= 62.2$
$P_{secular}$	$= 67.4$
$P_{num.int}$	$= 66.5,$

where the subscripts (2, 4, 6 and 8) indicate the highest-order term taken into account. The result P_{a_2} was evaluated using the approximation (30), while $P_{secular}$ was obtained using the full secular theory (with $\dot{\theta}_0 = 0.000\,658\,57$). The 'true' period $P_{num.int}$ was estimated from direct numerical integration of the motion. This estimation is difficult and there is an uncertainty in the last significant figure given.

One can conclude that the pure harmonic oscillator approximation P_2 is not very accurate, while the full secular model differs rather little from the accurate (numerical integration) result. Furthermore, the result P_{a_2} looks reasonable but in fact this is an approximation for the P_2 result. The difference is more than 13 yr and thus (30) is not very useful for this case. This is not surprising because that expression was derived assuming a nearly planar system.

Note that the value of $\Psi = -1.438\,826\,08$ is not far from $-\pi/2$ which is the most stabilizing value. This qualitatively agrees with the fact that 2004 GU9 is just about in the middle of its stay as a QS. Fig. 10 illustrates the actual evolution of the angle Ψ during the QS-episode of this asteroid. The angle changes from about $324°$ to $215°$. Taking into account the periodicity (equal to π) with respect to Ψ, we can compute that in Fig. 6 this corresponds to Ψ-motion of $2.51 \rightarrow 0.61$ (radians). Thus the agreement of this with our theoretical considerations seems good. It should be stressed here that the expansion coefficients are not constant; changing somewhat even during a single θ-oscillation period, and over a longer time, the coefficients may even change their signs.

The different approximations for the θ-potential are compared in Fig. 11 using 2004 GU9 once more as an example. Our first-order approximate perturbing function $R \approx \frac{m_1}{a_1}\langle U \rangle$ always gives a θ-potential that is completely symmetrical. This is the thin curve
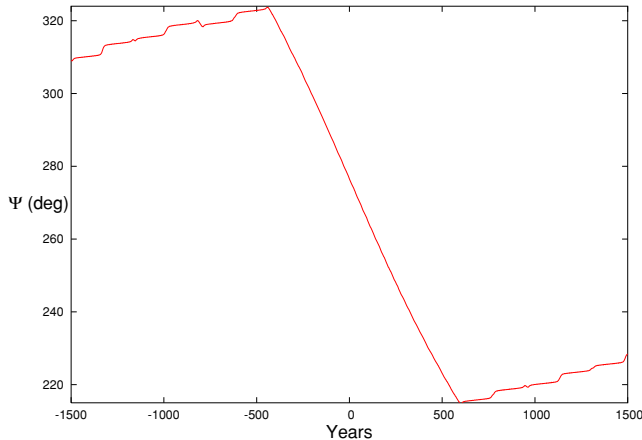
Figure 10. Evolution of Ψ during the stability period of 2004 GU9. One notes that 2004 GU9 is at the halfway point of its stay in the QS-motion.
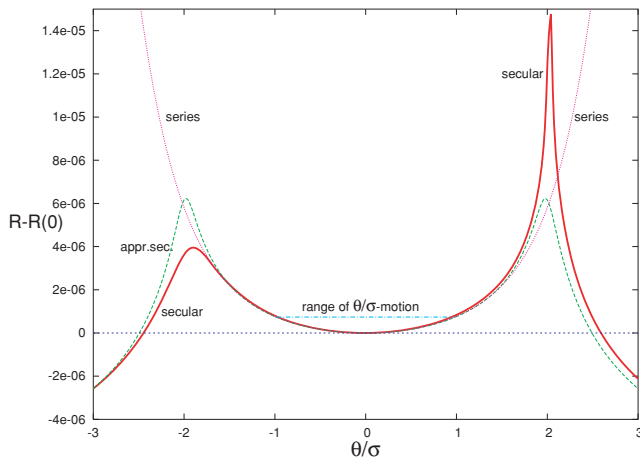

Figure 11. Comparison of $R(T) - R(0)$ for 2004 GU9 in different approximations (as earlier $T = \theta/\sigma$). The symmetrical curve marked 'appr.sec.' was obtained as $\frac{m_1}{a_1}\langle U \rangle$, the equally symmetrical curve 'series' represents the function in equation (46) and the thick line 'secular' is the full secular θ potential (equation (36). One notes the clear asymmetry of this best approximation.

marked 'appr.sec.' in the figure. It approximates rather well the more accurate ('secular') function for $T = \theta/\sigma < 2$. The power-series expansion (46) is also good for not too large T. However, the full secular theory (thick line marked 'secular' in the figure) behaves somewhat differently: there is a clear asymmetry, which allows an asteroid to enter the QS-state (from a horseshoe orbit), climbing over the lower side of the potential wall and then bouncing back from the higher wall to re-enter the horseshoe orbit. This requires a sufficiently high value of the semimajor axis difference which defines the θ-energy through the expression for $\dot\theta$.

In the figure the line marked 'range of motion' indicates the value by which the constant of integration R_x exceeds the near-zero minimum of the potential. This therefore also indicates the range of allowed oscillation of θ.

The amount of asymmetry depends on the eccentricities, inclinations and the mutual orientations of the orbits. Since the first-order approximation does not give any asymmetry, one must use that theory with care and, preferably, rely on the full secular theory.
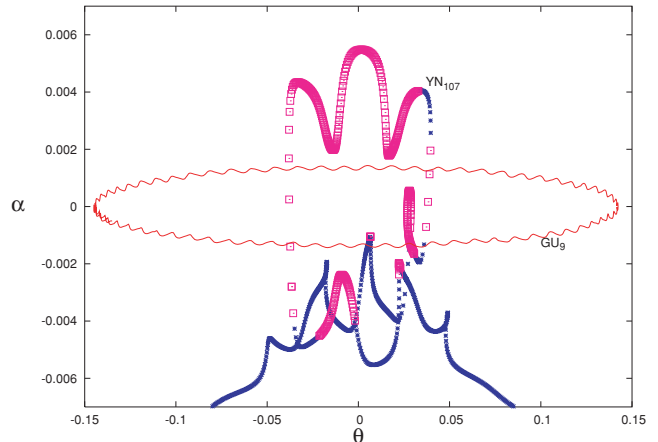
Figure 12. The behaviour of α as function of θ in the motions of 2003 YN107 (red squares and blue dots) and 2004 GU9 (line). For details see the text.

Fig. 12 illustrates the relations between θ and α for the two asteroids 2003 YN107 and 2004 GU9. These data were obtained by accurate numerical integration of the real system. The big (red) squares represent points where the algorithm to compute the period gives a result for 2003 YN107 (and that is about correct), while the small (blue) dots are plotted where the minima and maxima for θ were not found (i.e. the θ-energy was above the smaller of the maxima in the potential.) Thus in this case the theory does not apply reliably. This is because the asteroid in question is only just about in the domain of QS state of motion and the short period variations in α are comparable to the secular ones.

The narrow elliptic curve (with some small oscillations on it) gives the (θ, α) relation for 2004 GU9. This behaviour is very similar to that illustrated in Fig. 5 and the theory works well here.

6 CONCLUSIONS

We have presented a fairly simple analytical theory of QS motion and have given an algorithm useful in practice to detect asteroids in QS motion. A straightforward numerical example applied to a real asteroid has been provided.

The full secular theory suggests that the leading order analytical theory is not sufficient. In particular, it does not reveal the asymmetry of the effective θ-potential at all. Thus the full secular theory, even if it cannot be written in terms of elementary functions, must be preferred. The numerical effort to obtain the results from that theory is, however, relatively minor and does not form a serious obstacle.

The main results concerning stability are as follows.

(i) Permanently stable QS-motion is possible only for small enough inclination, that is, (approximately) for

$$I < |e - e_1|.$$

Such primordial QS-asteroids are, however, not yet known.

(ii) For large enough inclination

$$I > |e - e_1|,$$

suitable relative semimajor axis difference α and appropriate orbit orientation, a temporary capture to QS-motion can take place.

(iii) For the capture, the orientation condition may be written as

$$\Psi \approx k\pi, k = 0, 1, 2, \ldots$$

The length of the QS-period depends in a fairly complicated way on the orbital elements. As the examples show, this can vary from a brief episode lasting only a few years to a time interval of a thousand years.

(iv) One must be aware of that the theory does not always give a definitive answer for whether an asteroid is in QS-motion or not. Thus, especially if the result is affirmative, one must carefully check if the situation is a 'just about' one.

Whereas a complete theory of QS motion remains elusive, we hope that our results prove useful in the search of QSs.

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